Bisexual Branching Diffusions

by

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Abstract

We study the limiting behaviour of large systems of two types of Brownian particles undergoing bisexual branching. Particles of each type generate individuals of both types, and the respective branching law is asymptotically critical for the two-dimensional system, while being subcritical for each individual population.

The main result of the paper is that the limiting behaviour of suitably scaled sums and differences of the two populations is given by a pair of measure and distribution valued processes which, together, determine the limit behaviours of the individual populations.

Our proofs are based on the martingale problem approach to general state space processes. The fact that our limit involves both measure and distribution valued processes requires the development of some new methodologies of independent interest.

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Introduction

In this paper we shall study the limiting behaviour of large systems of two populations of Brownian particles undergoing bisexual branching. As an example, consider the following system:

Assume that, at time $t = 0$, $n \geq 1$ particles of two types (type 1 and type 2) are located in R^d and begin moving as independent Brownian motions. Each particle of type i, $i = 1, 2$, dies, independently of the others, after an exponential time with mean $1/n$. Assume that, at the time of death, each particle is replaced, on average, by one particle of each type, with overall probability $\frac{1}{2}$, and, with probability $\frac{1}{2}$, by nothing. The replacement particles, if there are any, perform independent Brownian motions starting at the point of death of their parents, and this story of diffusion and (critical) branching continues until, with probability one, there are no particles left.

Let $Z_i^{(n)}(t, \cdot)$ be the measure valued processes describing the positions of the particles at time t. That is, let

$$Z_i^{(n)}(t, A) \;=\; \text{Number of particles of type } i \text{ in } A \text{ at time } t.$$

Our interest lies in establishing the joint behaviour of the $Z_i^{(n)}(t)$ in the infinite density, $n \to \infty$, limit.

This problem, in the absence of particle motion, has already been studied by Kurtz [5]. The addition of the particle motion, however, makes the problem considerably more complex, and rather interesting.

In order to describe a typical result, define the following two, rescaled, processes.

$$X^{(n)}(t) \equiv \frac{Z_1^{(n)}(t) + Z_2^{(n)}(t)}{n}, \qquad Y^{(n)}(t) \equiv \frac{Z_1^{(n)}(t) - Z_2^{(n)}(t)}{n}.$$

We shall study the limiting behaviour of $X^{(n)}$ and $Y^{(n)}$ as $n \to \infty$. This, clearly, will tell us about the limiting behaviour of the $Z_i^{(n)}$. What we shall find is that while $X^{(n)}$ has an interesting nontrivial limit, $Y^{(n)}$ converges to zero. That is, in the infinite density limit the proportions of particles of each type are identical.

This, naturally, leads one to try to re-rescale the difference $Y^{(n)}$, so as to obtain a fluctuation result describing the rate at which balance between the two populations is achieved. It turns out that this is best done via a third process, defined by

$$W^{(n)}(t) \equiv Y^{(n)}(t) + \int_0^t nY^{(n)}(s)ds.$$

We shall show that $W^{(n)}$ has a nice limit, as $n \to \infty$, as a (Schwartz) distribution valued process, and that the convergence is joint with that of $X^{(n)}$.

The main result of the paper, Theorem 3.3, gives the details of this convergence, under a more general setup than that just described.

Since our study involves both measure and distribution-valued processes we shall, unfortunately, need to start with some technical results about weak convergence for measure cross distribution valued processes. Some of these results should be of independent interest. This is Section 1 of the paper.

In Section 2 we begin describing our system in detail, and describe also the previous result of Kurtz [5] noted above. Section 3 contains the main result of the paper, and Section 4 is devoted to proofs. There is an appendix containing the proof of the existence and uniqueness of the solution a particular non-linear evolution equation which is used in the proofs of Section 4.

1 Preliminaries, weak convergence

In this section we give a brief introduction to the martingale problem approach to general state space process. We start with some notation and definitions. Let E be topological space, $B(E)$ ($\overline{C}(E)$) be the set of bounded (bounded continuous) Borel measurable functions on E. Denote by $\mathcal{B}(E)$ the σ-algebra of Borel subsets of E and by $\mathcal{P}(E)$ the set of Borel probability measures on E. Let A be subset (not necessary linear) of $B(E) \times B(E)$.

Definition 1.1 *By a solution of the martingale problem for A we mean a measurable stochastic process X with values in E defined on some probability space $(\Omega, \mathcal{F}, P)$, such that for each $(f,g) \in A$ the process*

$$f(X(t)) - \int_0^t g(X(s))\,ds \tag{1.1}$$

is a martingale with respect to the filtration

$$\tilde{\mathcal{F}}_t^X = \mathcal{F}_t^X \vee \sigma\left(\int_0^t h(X(u))\,du : \ s \le t,\ h \in B(E)\right)$$

where $\mathcal{F}_t^X \equiv \sigma(X(s) : \ s \le t)$.

Note that if X is a right continuous process then $\tilde{\mathcal{F}}_t^X = \mathcal{F}_t^X$.

Definition 1.2 *When an initial distribution $\mu \in \mathcal{P}(E)$ is specified, we say that a solution of the martingale problem for A is a solution of the martingale problem for (A, μ) if $PX(0)^{-1} = \mu$.*

Let $\{\mathcal{F}_t\}$ be a complete filtration. Let $\mathcal{L}$ be the space of progressive (i.e. $\{\mathcal{F}_t\}$-progressive) processes Y such that $\sup_{t \ge 0} E[|Y(t)|] < \infty$. Set

$$\|Y\| = \sup_{t \ge 0} E[|Y(t)|]. \tag{1.2}$$

We do not distinguish between $\mathcal{L}$ and the Banach space of equivalence classes in $\mathcal{L}$, determined by the norm (1.2). ($X \sim Y$ if $\|X - Y\| = 0$). We define the semigroup of operators $\mathcal{J}(s)$ on $\mathcal{L}$ by

$$(\mathcal{J}(s)Y)(t) = E[Y(t+s)|\mathcal{F}_t]. \tag{1.3}$$

Define

$$\mathcal{A} = \left\{(Y, Z) \in \mathcal{L} \times \mathcal{L} : \ Y(t) - \int_0^t Z(s)\,ds \text{ is } \mathcal{F}_t\text{-martingale}\right\}. \tag{1.4}$$

$\mathcal{A}$ is called the full generator of $\mathcal{J}(s)$.

Let $\mathcal{G}(D_E[0,\infty))$ be the σ-algebra generated by the simple cylindrical subsets of $D_E[0,\infty)$; i.e.

$$\mathcal{G}(D_E[0,\infty)) = \sigma\left((\pi_{t_1,\ldots,t_m})^{-1}(\mathcal{B}(E))^{\otimes m} \mid t_1,\ldots,t_m \in [0,\infty),\ m \in \mathcal{N}\right),$$

where for points $t_1, t_2, \ldots, t_m$ in R_+, the projection $\pi_{t_1,t_2,\ldots,t_m}$: $D_E[0,\infty) \longrightarrow E^m$ is defined by

$$\pi_{t_1,t_2,\ldots,t_m}(x) = (x(t_1), x(t_2), \ldots, x(t_m)),\ \forall x \in D_E[0,\infty).$$

When

$$\mathcal{G}(D_E[0,\infty)) = \mathcal{B}(D_E[0,\infty)) \tag{1.5}$$

we shall say that E has equivalent Borel and cylindrical σ-algebras.

We now give an analogue of the Theorem 4.8.10 [5] without the assumption that the space E is metric. The result will be crucial for our later needs, as our main result will, ultimately, follow directly by checking that the conditions of this result are satisfied.

Theorem 1.3 *Let (E,τ) be a completely regular topological space with equivalent Borel and cylinder σ-algebras on $D_E[0,\infty)$. For each $n \geq 1$ let $\{\mathcal{F}_t^n\}$ be a complete filtration, and let $A \subset \overline{C}(E) \times \overline{C}(E)$ and $\nu \in \mathcal{P}(E)$. Suppose X_n, $n = 1,2,\ldots$, is a $\mathcal{F}_t^n$-adapted process with sample paths in $D_E[0,\infty)$, $\{X_n\}$ is relatively compact, and $PX_n(0)^{-1} \Longrightarrow \nu$, as $n \to \infty$. Suppose furthermore that for each $(f,h) \in A$ and $T > 0$, there exist $(\xi_n, \varphi_n) \in \mathcal{A}_n$, (where $\mathcal{A}_n$ is defined as at (1.4) but for $\mathcal{F}_t^n$ martingales), such that the following three conditions hold:*

$$\sup_n \sup_{s \leq T} E\left[|\xi_n(s)|\right] < \infty, \tag{1.6}$$

$$\sup_n \sup_{s \leq T} E\left[|\varphi_n(s)|\right] < \infty, \tag{1.7}$$

$$\lim_{n\uparrow\infty} E\left[|\xi_n(s) - f(X_n(t))|\right] = \lim_{n\uparrow\infty} E\left[|\varphi_n(s) - h(X_n(t))|\right] = 0. \tag{1.8}$$

Then

(a) *Each limit point of $\{X_n\}$ is a solution of the $D_E[0,\infty)$ martingale problem for (A,ν).*

(b) *If, in addition, we assume that the $D_E[0,\infty)$ martingale problem for (A,ν) has at most one solution, then $X_n \Longrightarrow X$, as $n \to \infty$, where X is the unique solution in $D_E[0,\infty)$ of the martingale problem for (A,ν).*

Proof The proof that each limit point of $\{X_n\}$ is a solution of the martingale problem for (A,ν) will be analogous to the proof of Theorem 4.8.10 [5].

Let Y be a limit point of $\{X_n\}$. Let $(f,h) \in A$ and $T > 0$, and let (ξ_n, φ_n) satisfy conditions (1.6 – 1.8). Let $k \geq 0$, $0 \leq t_1 < t_2 < \ldots < t_k \leq t < t+s < T$. Since $(\xi_n, \varphi_n) \in \widehat{\mathcal{A}}_n$, it follows that

$$E\left[\left(\xi_n(t+s) - \xi_n(t) - \int_t^{t+s} \varphi_n(u)\,du\right) \prod_{i=1}^k h_i(X_n(t_i))\right] = 0, \tag{1.9}$$

for all $h_1, \ldots, h_k \in B(E)$, $n \in \mathcal{N}$, which implies that

$$E\left[\left(\xi_n(t+s) - f(X_n(t+s)) - (\xi_n(t) - f(X_n(t))) - \int_t^{t+s} \varphi_n(u) - h(X_n(u))\, du\right) \prod_{i=1}^{k} h_i(X_n(t_i))\right]$$

$$+ E\left[\left(f(X_n(t+s)) - f(X_n(t)) - \int_t^{t+s} h(X_n(u))\, du\right) \prod_{i=1}^{k} h_i(X_n(t_i))\right] = 0, \ \forall n.$$

Taking into account condition (1.8), and that $h_1, \ldots, h_k \in B(E)$, we have that

$$\lim_{n\uparrow\infty} E\left[\left(f(X_n(t+s)) - f(X_n(t)) - \int_t^{t+s} h(X_n(u))\, du\right) \prod_{i=1}^{k} h_i(X_n(t_i))\right] = 0 \tag{1.10}$$

and hence

$$E\left[\left(f(Y(t+s)) - f(Y(t)) - \int_t^{t+s} h(Y(u))\, du\right) \prod_{i=1}^{k} h_i(Y(t_i))\right] = 0. \tag{1.11}$$

Thus we have that Y is a solution of the martingale problem for (A, ν), and part (a) is proved.

By (a) and (b) we have that the martingale problem for (A, ν) has a unique solution. This means that the finite-dimensional distributions of all limit points of $\{X_n\}$ coincide. Since the Borel and cylinder σ-algebras are equivalent, the finite-dimensional distributions determine the probability measure on $D_E[0, \infty)$. By this argument, together with relative compactness, we get that each subsequence of $\{X_n\}$ contains a further subsequence which converges to the unique solution of the martingale problem for (A, ν). By Theorem 2.3 [2] we are done. ∎

In order to apply (b) of Theorem 1.3 we need to know how to determine uniqueness for solutions of martingale problems for processes with values in arbitrary topological spaces.

Theorem 1.4 *Let (E, τ) be an arbitrary topological space, and let $A \subset B(E) \times B(E)$. Suppose that for each $\mu \in \mathcal{P}(E)$ any two solutions X, Y of the martingale problem for (A, μ) have the same one-dimensional distributions; i.e. for each $t > 0$,*

$$P\{X(t) \in \Gamma\} = P\{Y(t) \in \Gamma\}, \quad \Gamma \in \mathcal{B}(E). \tag{1.12}$$

Then any two solutions of the martingale problem for (A, μ) have the same finite-dimensional distributions; i.e. (1.12) implies uniqueness on the cylinder σ-algebra. If X is a solution of the martingale problem for (A, μ) with respect to the filtration $\mathcal{F}_t$, then

$$E[f(X(s+t)) \,|\, \mathcal{F}_s] = E[f(X(s+t)) \,|\, X(s)] \tag{1.13}$$

for all $f \in \mathcal{B}(E)$ and $s, t \geq 0$.

Proof The proof is completely analogous to the proof of Theorem 4.4.2 of [5].

Corollary 1.5 *Let (E, τ) be a completely regular topological space with equivalent Borel and cylinder σ-algebras on $D_E[0, \infty)$. Let $A \in B(E) \times B(E)$. Suppose that for each $\mu \in \mathcal{P}(E)$ any two solutions X, Y of the martingale problem for (A, μ) with sample paths in $D_E[0, \infty)$ satisfy (1.12) for each $t \geq 0$. Then, for each $\mu \in \mathcal{P}(E)$, any two solutions of the martingale problem for (A, μ) with sample paths in $D_E[0, \infty)$ have the same distributions on $D_E[0, \infty)$.*

Proof The fact that X and Y have the same finite-dimensional distributions is a part of Theorem 1.4. Since the Borel and cylinder σ-algebras are equivalent the finite-dimensional distributions of X and Y determine their distributions on $D_E[0,\infty)$, and we are done. ∎

Let $M_F(R^d)$ denote the space of finite measures on $(R^d, \mathcal{B}(R^d))$ endowed with the topology of weak convergence. For $\mu \in M_F(R^d)$ and $f \in \overline{C}(R^d)$, let

$$\langle f, \mu \rangle \equiv \int f \, d\mu. \tag{1.14}$$

Denote by $S(R^d)$ the (Schwartz) space of rapidly decreasing functions on R^d, and by $S'(R^d)$ the topological dual of $S(R^d)$, the space of tempered distributions. We endow $S'(R^d)$ with the strong topology.

For reasons that will become clear later we now need to study stochastic processes taking values in $M_F(R^d) \times S'(R^d)$. For simplicity denote $M_F(R^d) \times S'(R^d)$ by $M_F \times S'$. Recall that in the theory of weak convergence of processes in $D_E[0,\infty)$, where E is complete separable metric space, the basic equivalence (1.5) between the Borel and cylindrical σ-algebras is of crucial importance. It is not, however, clear that it carries over to the case of $E = M_F \times S'$, which is what we shall require. The general problem of determining which spaces E do possess equivalent Borel and cylindrical σ-algebras has been studied in some detail by Jakubowski [7]. Following his techniques, we shall establish

Theorem 1.6 *The space $D_{M_F \times S'}[0,\infty)$ has equivalent Borel and cylinder σ-algebras.*

Before the proof of this Theorem we note the following lemma from [7].

Lemma 1.7 **(i)** *Suppose that the completely regular topological space (E,τ) has the following two properties:*

- **(1)** *Compact subsets of E are metrizable.*
- **(2)** *There exists a sequence of $\{K_n\}$ of compact subsets of E such that for every $x \in D_E[0,1]$ one can find K_n containing the set $\hat{x} = \{x(t) | t \in [0,1]\}$.*

Then

$$\mathcal{G}(D_E[0,1]) = \mathcal{B}(D_E[0,1]). \tag{1.15}$$

(ii) *If a completely regular linear topological space E satisfies (1.15) it also satisfies (1.5).*

Proof of Theorem 1.6 First let us prove that $M_F \times S'$ generates equivalent Borel and cylinder σ-algebras for $D_{M_F \times S'}[0,1]$.

We know that M_F is a separable metric space, hence it is homeomorphic to a subset of R^∞. The space R has the properties (1) and (2) from Lemma 1.7, and by Proposition 5.3 [7] S' also satisfies these conditions. Hence by Corollary 2.8 [7] $R^\infty \times S'$ has the property (1.15) and by Theorem 2.1 [7] $A \times S'$ also has this property for each subset A of R^∞. By Jakubowski's Theorem 1.3 the topology on $D_E[0,1]$ depends only on the topology τ on E, consequently the property (1.15) is preserved under homeomorphism. Thus, by a homeomorphism of M_F to some subset of R^∞, we have that (1.15) is satisfied by $M_F \times S'$ and hence by part (ii) of Lemma 1.7 $D_{M_F \times S'}[0,\infty)$ has equivalent Borel and cylinder σ-algebras, as required. ∎

We need some additional properties of the space $M_F \times S'$.

Lemma 1.8 (i)

$$\mathcal{B}(M_F \times S') = \mathcal{B}(M_F) \times \mathcal{B}(S').$$

(ii) *Let $N_1 \subset \overline{C}(M_F)$ and $N_2 \subset \overline{C}(S')$ be separating for $\mathcal{P}(M_F)$ and $\mathcal{P}(S')$ respectively. Then*

$$N = \{f_1 f_2 : f_1 \in N_1 \cup 1,\ f_2 \in N_2 \cup 1\}$$

is separating for $\mathcal{P}(M_F \times S')$.

Proof (i) It is sufficient to show that

$$\mathcal{B}(M_F \times S') \subset \mathcal{B}(M_F) \times \mathcal{B}(S'). \tag{1.16}$$

Let A be a Borel subset of $M_F \times S'$. By Proposition 5.3 [7] we have that

$$A = \bigcup_{n \in \mathcal{N}} A \cap (M_F \times K_n), \tag{1.17}$$

where K_n are compact in S' and metrizable. Each compact metric space is separable, and M_F is a separable metric space, hence by [2], p.225 we have that

$$\mathcal{B}(M_F \times K_n) = \mathcal{B}(M_F) \times \mathcal{B}(K_n),\ \forall n. \tag{1.18}$$

From the definition of the relative topology in $M_F \times K_n$ we have

$$\begin{aligned} \mathcal{B}(M_F \times K_n) &\equiv \sigma(B \cap (M_F \times K_n),\ B \in \tau) \\ &= \{B \cap (M_F \times K_n),\ B \in \mathcal{B}(M_F \times S')\}, \end{aligned} \tag{1.19}$$

where τ is the topology on $M_F \times S'$, and the last equality follows by [3], Theorem 10.1. By choice, $A \in \mathcal{B}(M_F \times S')$, and so

$$A \cap (M_F \times K_n) \in \mathcal{B}(M_F \times K_n).$$

Thus, by (1.18),

$$A \cap (M_F \times K_n) \in \mathcal{B}(M_F) \times \mathcal{B}(K_n).$$

By the same arguments as in (1.19) we get

$$\mathcal{B}(K_n) = \{B \cap K_n,\ B \in \mathcal{B}(S')\},$$

so that

$$\mathcal{B}(K_n) \subset \mathcal{B}(S'),\ \forall n,$$

and

$$A \cap (M_F \times K_n) \in \mathcal{B}(M_F) \times \mathcal{B}(S'),\ \forall n.$$

By (1.17) we obtain

$$A \in \mathcal{B}(M_F) \times \mathcal{B}(S'),$$

and we are done.

(ii) The proof of this part is completely analogous to that of Proposition 3.4.6 [5]. ∎

Let $C_l(R^d)$ denote the set of continuous functions with limit at infinity. In general, if F is a set of functions on R^d, write F_+ for $\{f \in F : \inf_{R^d} f(x) > 0\}$.

Corollary 1.9 *The set of functions*

$$\left\{F \in \overline{C}(M_F \times S') : \ F_{f_1,f_2}(\mu_1,\mu_2) \equiv \exp\{\langle f_1,\mu_1\rangle + i\,\langle f_2,\mu_2\rangle\}\right\},$$

where $f_1 \in C_l(R^d)_+$, and $f_2 \in S(R^d)$, is separating on $\mathcal{P}(M_F \times S')$.

Proof By [4], Theorem 3.2.6, the set of functions $\{\exp\{\langle f,\cdot\rangle\},\ f \in C_l(R^d)_+\}$ is separating on $\mathcal{P}(M_F)$. By [6], Theorem 3.2, the probability law of X, where X is a random distribution, is uniquely determined by the characteristic functional

$$C_X(f) = E\left[\exp\{i\,\langle f, X\rangle\}\right],\ f \in S(R^d),$$

and hence the set of functions $\exp\{i\,\langle f_2,\cdot\rangle\}$ is separating on $\mathcal{P}(S')$. The result then follows by Lemma 1.8, in spite of the fact that $1 \notin \{\exp\{\langle f,\cdot\rangle\},\ f \in C_l(R^d)_+\}$, since there exists $\{f_n\} \in C_l(R^d)_+$, such that

$$\text{bp-}\lim_{n\to\infty} \exp\{\langle f,\cdot\rangle\} = 1. \tag{1.20}$$

∎

2 Bisexual branching without diffusion

In this section we shall, briefly, describe some results of Kurtz [5], which correspond to a generalization of the bisexual branching system described in the Introduction, but for which the "particles" perform no motion. In the following section we shall extend this model to the one of interest to us, but it is worthwhile, at this stage, to look at the simpler case.

Kurtz [5] considered a system made up of two types of particles. Each particle lives for an exponentially distributed lifetime with parameter λ_1 or λ_2, depending on its type. If a type 1 particle dies it gives rise to offspring of types 1 and 2 with the number of offspring distributed as (γ_1,γ_2). Similarly, if a type 2 particle dies, it gives rise to offspring of both types distributed as (ψ_1,ψ_2). Assume that $E[\gamma_i^3] < \infty$, $E[\psi_i^3] < \infty$, $i = 1,2$. Define

$$m_{1j} \equiv E[\gamma_j], \qquad m_{2j} \equiv E[\psi_j], \qquad j = 1,2.$$

Let Z_j be the number of type j particles alive at time t, and set

$$Z(t) = (Z_1(t),\ Z_2(t)), \qquad t \geq 0. \tag{2.1}$$

Then Z is is a two-type Markov branching process. We shall assume that the process is critical and $m_{ij} > 0$ for all $i, j = 1, 2$. This implies that there exist vectors (ν_1, ν_2) and (ξ_1, ξ_2) and a real number $\eta > 0$ satisfying

$$\text{(2.2)} \qquad \begin{pmatrix} \lambda_1(m_{11}-1) & \lambda_1 m_{12} \\ \lambda_2 m_{21} & \lambda_2(m_{22}-1) \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = 0,$$

$$\text{(2.3)} \qquad \begin{pmatrix} \lambda_1(m_{11}-1) & \lambda_1 m_{12} \\ \lambda_2 m_{21} & \lambda_2(m_{22}-1) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = -\eta \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Taking $\nu_1, \nu_2 > 0$ and ξ_1 and ξ_2 will have opposite signs. Kurtz considers a sequence of processes $\{(Z_1^{(n)}, Z_2^{(n)})\}$ with initial population sizes $(Z_1^{(n)}(0), Z_2^{(n)}(0)) = ([nz_1], [nz_2])$, where (z_1, z_2) are fixed. Set

$$\text{(2.4)} \qquad X^{(n)}(t) \equiv \frac{\nu_1 Z_1^{(n)}(nt) + \nu_2 Z_2^{(n)}(nt)}{n},$$

$$\text{(2.5)} \qquad Y^{(n)}(t) \equiv \frac{\xi_1 Z_1^{(n)}(nt) + \xi_2 Z_2^{(n)}(nt)}{n}.$$

Then both $X^{(n)}(t)$ and $Y^{(n)}(t)\exp\{n\eta t\}$ are martingales [1]. We are interested in the limiting behaviour of $X^{(n)}(t)$ and $Y^{(n)}(t)$ as $n \to \infty$. As might be expected from the fact that $Y^{(n)}(t)\exp\{n\eta t\}$ is a martingale, $Y^{(n)}(t)$ converges to zero as $n \to \infty$, and so

$$\frac{Z_1^{(n)}(nt)}{n} \approx \frac{\xi_2 X^{(n)}(t)}{\nu_1\xi_2 - \nu_2\xi_1}, \quad \text{and} \quad \frac{Z_2^{(n)}(nt)}{n} \approx \frac{\xi_1 X^{(n)}(t)}{\nu_2\xi_1 - \nu_1\xi_2}.$$

Thus we have the rather surprising result that the limiting behaviour of $X^{(n)}$ gives the limiting behaviour of both $Z_1^{(n)}(nt)/n$ and $Z_2^{(n)}(nt)/n$ for $t > 0$.

The limiting behaviour of $Y^{(n)}$ is somewhat more delicate, and is best described in terms of a new process $W^{(n)}$ defined by

$$W^{(n)}(t) = Y^{(n)}(t) + \int_0^t n\eta Y^{(n)}(s)\,ds.$$

To state Kurtz's main theorem we require the random variables

$$\text{(2.6)} \qquad \begin{aligned} \tilde{\gamma}_1 &\equiv \nu_1(\gamma_1 - 1) + \nu_2\gamma_2\,, \quad \tilde{\gamma}_2 \equiv \xi_1(\gamma_1 - 1) + \xi_2\gamma_2, \\ \tilde{\psi}_1 &\equiv \nu_1\psi_1 + \nu_2(\psi_2 - 1)\,, \quad \tilde{\psi}_2 \equiv \xi_1\psi_1 + \xi_2(\psi_2 - 1). \end{aligned}$$

and the parameters $a_{ij}^1 = E[\tilde{\gamma}_i\tilde{\gamma}_j]$ and $a_{ij}^2 = E[\tilde{\psi}_i\tilde{\psi}_j]$. Then Kurtz [8] proved

Theorem 2.1 (a) *The sequence $\{(X^{(n)}, W^{(n)})\}$ converges in distribution to a R^2-valued diffusion (X, W) with generator*

$$\text{(2.7)} \qquad Af(x,w) = \frac{x}{2}\left(a_{11}f_{xx}(x,w) + 2a_{12}f_{x,w}(x,w) + a_{22}f_{ww}(x,w)\right)$$

where $a_{ij} = (\lambda_1\xi_2 a^1_{ij} - \lambda_2\xi_1 a^2_{ij})/(\nu_1\xi_2 - \xi_1\nu_2)$.

(b) *For each* $T > 0$,

$$\sup_{t\leq T} |Y^{(n)}(t) - Y^{(n)}(0)\exp\{-n\eta t\}| \to 0, \quad \textit{in probability, as } n \to \infty.$$

Consequently, for $0 < t_1 < t_2$, *it follows that* $\int_{t_1}^{t_2} n\eta Y^{(n)}(s)\,ds$ *converges in distribution to* $W(t_2) - W(t_1)$ *as* $n \to \infty$.

Our aim now is to add diffusion into the above model, and obtain a version of Theorem 2.1 incorporating the spatial motion.

3 Bisexual branching with diffusion

We start with a "typical" situation. Assume that at time $t = 0$ we have a number of particles of types 1 and 2 scattered throughout R^d, which begin diffusing as independent Brownian motions with generators $\frac{1}{2}\Delta$. Each particle of type i, $i = 1, 2$, dies, independently of the others, after an exponential time with parameter λ_i. Let p^i_{kl} be the probability that at death a particle of type i produces k offspring of type 1 and l offspring of type 2. For Borel $A \subset R^d$, set

$$Z_i(t, A) \;=\; \text{Number of particles of type } i \text{ in the set } A \text{ at time } t.$$

Then $Z_i(t) \in M_F$ for all $t \geq 0$. Consider the vector measure-valued process

$$\tilde{Z}(t) = (Z_1(t), Z_2(t)),$$

adapted to the filtration

$$\mathcal{F}_t \equiv \sigma\left(\tilde{Z}s :\; s \leq t\right).$$

We write

$$(3.1) \qquad |Z_i(t)| \;\equiv\; Z_i(t, R^d),\; i = 1, 2,$$

to denote the total mass processes, essentially equivalent to the population processes discussed in the previous section.

Take now a sequence $\tilde{Z}^{(n)}(t) = (Z_1^{(n)}(t),\, Z_2^{(n)}(t))$ of such processes with death intensities $\lambda_1^{(n)}$, $\lambda_2^{(n)}$, and offspring distributions $p_{kl}^{1(n)}$, $p_{kl}^{2(n)}$, adapted to the filtrations $\mathcal{F}_t^n = \sigma\left(\tilde{Z}^{(n)}(s) :\; s \leq t\right)$.

Let $({\gamma_1}^{(n)}, {\gamma_2}^{(n)})^T$ have joint distribution $p_{kl}^{1(n)}$, and $(\psi_1^{(n)}, \psi_2^{(n)})^T$ have joint distribution $p_{kl}^{2(n)}$. Set

$$m_{1j}^{(n)} \equiv E\left[{\gamma_j}^{(n)}\right], \quad m_{2j}^{(n)} \equiv E\left[\psi_j^{(n)}\right].$$

Let $(\nu_1^{(n)}, \nu_2^{(n)})^T$, $(\xi_1^{(n)}, \xi_2^{(n)})^T$ denote eigenvectors corresponding to the eigenvalues $\eta_1^{(n)}$, $-\eta_2^{(n)}$ of the matrix $\Lambda^{(n)}$, where

$$\Lambda^{(n)} \equiv \begin{pmatrix} \lambda_1^{(n)}(m_{11}^{(n)} - 1) & \lambda_1^{(n)} m_{12}^{(n)} \\ \lambda_2^{(n)} m_{21}^{(n)} & \lambda_2^{(n)}(m_{22}^{(n)} - 1) \end{pmatrix}. \tag{3.2}$$

Then we make the following regularity assumptions:

$$0 < m_{ij}^{(n)} < 1,\ \forall n, \qquad 0 < m_{ij} \equiv \lim_{n\uparrow\infty} m_{ij}^{(n)} < 1, \quad i,j = 1,2, \tag{3.3}$$

$$\lambda_i^{(n)} > 0,\ i = 1,2,\ \forall n,\ \text{ either } \lim_{n\uparrow\infty} \frac{\lambda_2^{(n)}}{\lambda_1^{(n)}} \text{ or } \lim_{n\uparrow\infty} \frac{\lambda_1^{(n)}}{\lambda_2^{(n)}} \text{ exists, } \lim_{n\uparrow\infty} \lambda_i^{(n)} = \infty,\ i = 1.2, \tag{3.4}$$

$$\lim_{n\uparrow\infty} \left[\left(m_{11}^{(n)} - 1\right)\left(m_{22}^{(n)} - 1\right) - m_{12}^{(n)} m_{21}^{(n)} \right] = 0, \tag{3.5}$$

$$\lim_{n\uparrow\infty} \left| \frac{\lambda_1^{(n)}\lambda_2^{(n)} \left(\left(m_{11}^{(n)} - 1\right)\left(m_{22}^{(n)} - 1\right) - m_{12}^{(n)} m_{21}^{(n)} \right)}{\lambda_1^{(n)}(1 - m_{11}^{(n)}) + \lambda_2^{(n)}(1 - m_{22}^{(n)})} \right| < \infty, \tag{3.6}$$

$$\sup_n E\left[\left(\gamma_j{}^{(n)}\right)^3\right] < \infty, \qquad \sup_n E\left[\left(\psi_j^{(n)}\right)^3\right] < \infty, \tag{3.7}$$

$$\sup_n \frac{\lambda_i^{(n)}}{n} < \infty,\ i = 1,2. \tag{3.8}$$

Simple but tedious calculations show that

$$\eta_1^{(n)} = -\frac{\lambda_1^{(n)}\lambda_2^{(n)} \left(\left(m_{11}^{(n)} - 1\right)\left(m_{22}^{(n)} - 1\right) - m_{12}^{(n)} m_{21}^{(n)} \right)}{\lambda_1^{(n)}(1 - m_{11}^{(n)}) + \lambda_2^{(n)}(1 - m_{22}^{(n)})} + o(1) \tag{3.9}$$

$$\eta_2^{(n)} = \lambda_1^{(n)}(1 - m_{11}^{(n)}) + \lambda_2^{(n)}(1 - m_{22}^{(n)}) + O(\eta_1^{(n)}). \tag{3.10}$$

Set

$$\eta_1 \equiv \lim_{n\uparrow\infty} \eta_1^{(n)}. \tag{3.11}$$

Under conditions (3.3) – (3.6) we obtain that $|\eta_1| < \infty$.

Despite the heavy notation above, things are are not as difficult as they seem. For example, if we return to the example of the Introduction, then we have $\lambda_1^{(n)} = \lambda_2^{(n)} = n$ and $m_{ij}^{(n)} = 1/2$, $i,j = 1,2$. It then follows that $\eta_1^{(n)} = 0$, $\eta_2^{(n)} = n$ and we can take $\nu_1^{(n)} = \nu_2^{(n)} = 1$, and $\xi_1^{(n)} = -1$, $\xi_2^{(n)} = 1$ for each $n > 0$.

The following technical lemma follows via straightforward algebra.

Lemma 3.1 *Under conditions (3.3) - (3.6) there exists a finite $N \geq 1$ and eigenvectors $(\nu_1^{(n)}, \nu_2^{(n)})$, $(\xi_1^{(n)}, \xi_2^{(n)})$ such that*

$$\inf_{n \geq N} \nu_i^{(n)} > 0, \tag{3.12}$$

$$\xi_1^{(n)} < 0, \quad \xi_2^{(n)} > 0, \quad \forall\, n \geq N, \qquad \lim_{n\uparrow\infty} \left|\xi_i^{(n)}\right| < \infty, \tag{3.13}$$

$$\lim_{n\uparrow\infty} \left[\nu_1^{(n)}\xi_2^{(n)} - \nu_2^{(n)}\xi_1^{(n)}\right] > 0. \tag{3.14}$$

Remark 3.2 *Without loss of generality we shall, henceforth, assume that $N = 1$ in (3.12).*

As in the case of Kurtz's simpler model, we need to define a few more additional random variables and coefficients before we can state our main result:

$$\begin{aligned} \tilde{\gamma}_1^n &\equiv \nu_1^{(n)}(\gamma_1{}^{(n)} - 1) + \nu_2^{(n)}\gamma_2{}^{(n)}\,, \quad \tilde{\gamma}_2^n \equiv \xi_1^{(n)}(\gamma_1{}^{(n)} - 1) + \xi_2^{(n)}\gamma_2{}^{(n)}, \\ \tilde{\psi}_1^n &\equiv \nu_1^{(n)}\psi_1^{(n)} + \nu_2^{(n)}(\psi_2^{(n)} - 1)\,, \quad \tilde{\psi}_2^n \equiv \xi_1^{(n)}\psi_1^{(n)} + \xi_2^{(n)}(\psi_2^{(n)} - 1). \end{aligned} \tag{3.15}$$

$$\begin{cases} a_{ij}^{(n)} \equiv \dfrac{\lambda_1^{(n)}\xi_2^{(n)}a_{ij}^{1(n)} - \lambda_2^{(n)}\xi_1^{(n)}a_{ij}^{2(n)}}{n\left(\nu_1^{(n)}\xi_2^{(n)} - \nu_2^{(n)}\xi_1^{(n)}\right)} \\[2ex] a_{ij}^{1(n)} \equiv E\left[\tilde{\gamma}_i^n\tilde{\gamma}_j^n\right], \quad a_{ij}^{2(n)} \equiv E\left[\tilde{\psi}_i^n\tilde{\psi}_j^n\right],\ i,j = 1,2. \end{cases} \tag{3.16}$$

Assume that $\lim_{n\uparrow\infty} a_{ij}^{(n)} < \infty$ exists (this is not a strong restriction given the results of the previous lemma and assumptions (3.3) - (3.8)) and define

$$a_{ij} \equiv \lim_{n\uparrow\infty} a_{ij}^{(n)}. \tag{3.17}$$

We are finally in a position to define the three measure valued processes that interest us. (Note that the second two of these are signed measures, or distributions.)

$$\begin{cases} X^{(n)}(t) \equiv \dfrac{\nu_1^{(n)}Z_1^{(n)}(t) + \nu_2^{(n)}Z_2^{(n)}(t)}{n}, \\[2ex] Y^{(n)}(t) \equiv \dfrac{\xi_1^{(n)}Z_1^{(n)}(t) + \xi_2^{(n)}Z_2^{(n)}(t)}{n}, \\[2ex] W^{(n)}(t) \equiv Y^{(n)}(t) + \displaystyle\int_0^t \eta_2^{(n)}Y^{(n)}(s)ds\,, \end{cases} \tag{3.18}$$

In general, let $\mathcal{D}(A)$ denote the domain of an operator A. Here is the main result of this paper:

Theorem 3.3 *Let $\{(X^{(n)}(0), W^{(n)}(0))\}$ have a limiting distribution $\nu \in \mathcal{P}(M_F \times S)$, and assume that $\{|X^{(n)}(0)|^2\}$ is uniformly integrable. Suppose that (3.3) - (3.8) hold and that $\lim_{n\to\infty} a_{ij}^{(n)}$ exists and is finite. Then $(X^{(n)}, W^{(n)}) \Longrightarrow (X, W)$, where $(X, W) \in C_{M_F \times S'}[0,\infty)$ is the unique solution in $D_{M_F\times S'}[0,\infty)$ of the following martingale problem for (A, ν):*

$$
\begin{aligned}
A = \Big\{ &\exp\{-\langle g_1, \mu_1\rangle + i\,\langle g_2, \mu_2\rangle\}, \\
&\exp\{-\langle g_1, \mu_1\rangle + i\,\langle g_2, \mu_2\rangle\}\left\langle \tfrac{1}{2}\left(-\Delta g_1 - 2\eta_1 g_1 + a_{11} g_1^2 - 2i a_{12} g_1 g_2 - a_{22} g_2^2\right), \mu_1\right\rangle : \\
&g_1 \in \tilde{\mathcal{D}}(\tfrac{1}{2}\Delta)_+ ,\ g_2 \in S(R^d)\Big\},
\end{aligned}
\tag{3.19}
$$

where a_{ij} and η_1 are as in (3.17) and (3.11) and $\tilde{\mathcal{D}}(\frac{1}{2}\Delta)_+ \equiv \mathcal{D}(\frac{1}{2}\Delta) \cap C_l(R^d)_+$.

Furthermore, for each $T > 0$ and $g_2 \in S(R^d)$,

$$
\sup_{t\le T}\left|\left\langle g_2, Y^{(n)}(t)\right\rangle - \left\langle g_2, Y^{(n)}(0)\right\rangle \exp\left\{-\eta_2^{(n)} t\right\}\right| \to 0, \quad \textit{in probability,}
\tag{3.20}
$$

as $n \to \infty$.

The rest of this paper is devoted to the proof of above theorem. The proof will rely on checking the conditions of Theorem 1.3.

4 Proofs

There are a number of steps involved in applying Theorem 1.3 in order to prove Theorem 3.3. The first of these lies in finding the approximations ξ_n and φ_n to $f(X^{(n)}, W^{(n)})$ and $h(X^{(n)}, W^{(n)})$, where $(f, h) \in A$. We do this in the following subsection, denoting the approximations, in Lemma 4.1, by $f^{(n)}$ and $h^{(n)}$.

Section 4.2 contains a sequence of preparatory lemmas that ultimately show that $(X^{(n)}, W^{(n)})$ satisfies a compact containment condition, required in Section 4.3, for showing that $(X^{(n)}, W^{(n)})$ is relatively compact in $D_{M_F\times S'}[0,\infty)$, a necessary requirement of Theorem 1.3.

The final requirement of Theorem 1.3, that the martingale problem (A, ν) have a unique solution, is established in Section 4.5 . This proof, itself, relies on a uniqueness result for a particular non-linear evolution equation, whose proof is relegated to an Appendix.

In Section 4.4 we show that the limit process, which, according to Theorem 3.3 and the previous steps is in $D_{M_F\times S'}[0,\infty)$, is actually continuous, and that (3.20) holds.

4.1 The martingale approximation

We need some simple notation which will be used later. If for sequence of random variables $\{\xi_n\}$ there exists a constant $C > 0$, independent of n, and another sequence $\{\psi_n\}$ such that

$$
|\xi_n| \le C\,|\psi_n|,\ \forall n
\tag{4.1}
$$

we shall write that $\xi_n = O(\psi_n)$. Denote by $o(1)$ a sequence of uniformly bounded random variables which converges to zero as $n \to \infty$.

Define $\widetilde{L}$ by:

$$(4.2) \qquad \widetilde{L} = \left\{ f \in C^{\infty}(R_+ \times R) : \lim_{x \longrightarrow \infty} \frac{\partial^{k+m} f(x,w)}{\partial^k x \partial^m w} x^n = 0, \ \forall\, k,n,m \geq 0 \right\}.$$

For simplicity we shall denote by f_x, f_w, f_{xx}, f_{ww}, f_{xw} the first and second order partial derivatives of $f \in \widetilde{L}$.

For each $f \in \widetilde{L}$, $g_1 \in \overline{C}(R^d)_+$ (where $\overline{C}(R^d)$ is the set of bounded continuous functions on R^d), and $g_2 \in S(R^d)$ define

$$(4.3) \qquad G_{f,g_1,g_2} : M_F \times S' \longrightarrow R$$

by

$$(4.4) \qquad G_{f,g_1,g_2}(\mu_1, \mu_2) = f\left(\langle g_1, \mu_1 \rangle, \langle g_2, \mu_2 \rangle\right), \ \forall (\mu_1, \mu_2) \in M_F \times S'.$$

Furthermore, define the operator $A_1 \in \overline{C}(M_F \times S') \times \overline{C}(M_F \times S')$ with domain

$$(4.5) \qquad \mathcal{D}(A_1) = \left\{ G_{f,g_1,g_2} : f \in \widetilde{L}, \ g_1 \in \mathcal{D}\left(\tfrac{1}{2}\Delta\right)_+, \ g_2 \in S\left(R^d\right) \right\}$$

by

$$\begin{aligned}
(4.6) \qquad A_1 G_{f,g_1,g_2}(\mu_1,\mu_2) &= f_x\left(\langle g_1,\mu_1\rangle, \langle g_2,\mu_2\rangle\right)\langle \eta_1 g_1, \mu_1\rangle \\
&\quad + \tfrac{1}{2} f_{xx}\left(\langle g_1,\mu_1\rangle, \langle g_2,\mu_2\rangle\right)\left\langle a_{11}(g_1)^2, \mu_1\right\rangle \\
&\quad + f_{xw}\left(\langle g_1,\mu_1\rangle, \langle g_2,\mu_2\rangle\right)\langle a_{12} g_1 g_2, \mu_1\rangle \\
&\quad + \tfrac{1}{2} f_{ww}\left(\langle g_1,\mu_1\rangle, \langle g_2,\mu_2\rangle\right)\left\langle a_{22}(g_2)^2, \mu_1\right\rangle \\
&\quad + f_x\left(\langle g_1,\mu_1\rangle, \langle g_2,\mu_2\rangle\right)\left\langle \tfrac{1}{2}\Delta g_1, \mu_1\right\rangle,
\end{aligned}$$

where the a_{ij} are defined by (3.17).

Lemma 4.1 *For all $G_{f,g_1,g_2} \in \mathcal{D}(A_1)$ there exists $(f_n, h_n) \in \mathcal{A}_n$, such that :*

$$\begin{aligned}
f_n(t) &= G_{f,g_1,g_2}(X^{(n)}, W^{(n)}) + O\left(\left(\eta_2^{(n)}\right)^{-1} |X^{(n)}(t)|\right), \\
h_n(t) &= A_1 G_{f,g_1,g_2}(X^{(n)}, W^{(n)}) + O\left(n^{-1}|X^{(n)}(t)|\right) + O\left(\left(\eta_2^{(n)}\right)^{-1} |X^{(n)}(t)|^2\right) \\
&\quad + O\left(\left(\eta_2^{(n)}\right)^{-1} |X^{(n)}(t)|\right) + o(1).
\end{aligned}$$

Proof Set

$$\tilde{f}^{(n)}(t) = f\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right).$$

To find $\tilde{h}^{(n)}$ so that $(\tilde{f}^{(n)}, \tilde{h}^{(n)}) \in \mathcal{A}_n$, we calculate $\lim_{\epsilon \to 0} \epsilon^{-1} E\left[\tilde{f}^{(n)}(t+\epsilon) - \tilde{f}^{(n)}(t) | \mathcal{F}_t^n\right]$ and obtain

$$\begin{aligned}
(4.7) \quad \tilde{h}^{(n)}(t) = \Big\langle &\tfrac{1}{2} F^{1(n)}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right) \\
&+ \lambda_1^{(n)} E_{\tilde{\gamma}^n}\Big[f\left(\left\langle g_1, X^{(n)}(t)\right\rangle + \frac{\tilde{\gamma}_1^n}{n} g_1, \left\langle g_2, W^{(n)}(t)\right\rangle + \frac{\tilde{\gamma}_2^n}{n} g_2\right) \\
&- f\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\Big], Z_1^{(n)}(t)\Big\rangle \\
+ \Big\langle &\tfrac{1}{2} F^{2(n)}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right) \\
&+ \lambda_2^{(n)} E_{\tilde{\psi}^n}\Big[f\left(\left\langle g_1, X^{(n)}(t)\right\rangle + \frac{\tilde{\psi}_1^n}{n} g_1, \left\langle g_2, W^{(n)}(t)\right\rangle + \frac{\tilde{\psi}_2^n}{n} g_2\right) \\
&- f\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\Big], Z_2^{(n)}(t)\Big\rangle \\
+ f_w &\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right) \left\langle \eta_2^{(n)} g_2, Y^{(n)}(t)\right\rangle \text{ a.s.}
\end{aligned}$$

where

$$\begin{aligned}
(4.8) \quad F^{1(n)}(x, w)(\cdot) &= f_{xx}(x, w)\left(\frac{\nu_1^{(n)}}{n}\right)^2 \sum_{i=1}^{d} (\partial_i g_1(\cdot))^2 \\
&+ 2 f_{xw}(x, w) \frac{\nu_1^{(n)} \xi_1^{(n)}}{n^2} \sum_{i=1}^{d} \partial_i g_1(\cdot) \partial_i g_2(\cdot) \\
&+ f_{ww}(x, w)\left(\frac{\xi_1^{(n)}}{n}\right)^2 \sum_{i=1}^{d} (\partial_i g_2(\cdot))^2 \\
&+ f_x(x, w) \frac{\nu_1^{(n)}}{n} \Delta g_1(\cdot) \\
&+ f_w(x, w) \frac{\xi_1^{(n)}}{n} \Delta g_2(\cdot),
\end{aligned}$$

$$\begin{aligned}
(4.9) \quad F^{2(n)}(x, w)(\cdot) &= f_{xx}(x, w)\left(\frac{\nu_2^{(n)}}{n}\right)^2 \sum_{i=1}^{d} (\partial_i g_1(\cdot))^2 \\
&+ 2 f_{xw}(x, w) \frac{\nu_2^{(n)} \xi_2^{(n)}}{n^2} \sum_{i=1}^{d} \partial_i g_1(\cdot) \partial_i g_2(\cdot)
\end{aligned}$$

$$+ f_{ww}(x,\, w) \left(\frac{\xi_2^{(n)}}{n} \right)^2 \sum_{i=1}^{d} (\partial_i g_2(\cdot))^2$$

$$+ f_x(x,\, w) \frac{\nu_2^{(n)}}{n} \Delta g_1(\cdot)$$

$$+ f_w(x,\, w) \frac{\xi_2^{(n)}}{n} \Delta g_2(\cdot).$$

Expanding f in a Taylor series about $(\langle g_1, X^{(n)}(t)\rangle, \langle g_2, W^{(n)}(t)\rangle)$, we have

$$\begin{aligned}
(4.10)\quad \tilde{h}^{(n)}(t) = \Big\langle & \tfrac{1}{2} F^{1(n)} \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) \\
& + \frac{\lambda_1^{(n)}}{n} \Big[f_x \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) E\left[\tilde{\gamma}_1^n\right] g_1 \\
& + f_w \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) E\left[\tilde{\gamma}_2^n\right] g_2 \\
& + \frac{1}{2n} f_{xx} \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) E\left[(\tilde{\gamma}_1^n)^2\right] (g_1)^2 \\
& + \frac{1}{2n} f_{ww} \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) E\left[(\tilde{\gamma}_2^n)^2\right] (g_2)^2 \\
& + \frac{1}{n} f_{xw} \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) E\left[\tilde{\gamma}_1^n \tilde{\gamma}_2^n\right] g_1 g_2 \Big],\ Z_1^{(n)}(t) \Big\rangle \\
+ \Big\langle & \tfrac{1}{2} F^{2(n)} \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) \\
& + \frac{\lambda_2^{(n)}}{n} \Big[f_x \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) E\left[\tilde{\psi}_1^n\right] g_1 \\
& + f_w \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) E\left[\tilde{\psi}_2^n\right] g_2 \\
& + \frac{1}{2n} f_{xx} \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) E\left[\left(\tilde{\psi}_1^n\right)^2\right] (g_1)^2 \\
& + \frac{1}{2n} f_{ww} \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) E\left[\left(\tilde{\psi}_2^n\right)^2\right] (g_2)^2 \\
& + \frac{1}{n} f_{xw} \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) E\left[\tilde{\psi}_1^n \tilde{\psi}_2^n\right] g_1 g_2 \Big],\ Z_2^{(n)}(t) \Big\rangle \\
+ f_w & \left(\left\langle g_1, X^{(n)}(t) \right\rangle, \left\langle g_2, W^{(n)}(t) \right\rangle \right) \left\langle \eta_2^{(n)} g_2, Y^{(n)}(t) \right\rangle + n^{-1} O\left(\left| X^{(n)}(t) \right| \right).
\end{aligned}$$

The error is $O(n^{-1}|X^{(n)}(t)|)$, since $n^{-1}|Z_i^{(n)}(t)|$ is bounded by a constant times $|X^{(n)}(t)|$. Recalling that

$$Z_1^{(n)}(t) = \frac{\xi_2^{(n)}X^{(n)}(t) - \nu_2^{(n)}Y^{(n)}(t)}{\nu_1^{(n)}\xi_2^{(n)} - \nu_2^{(n)}\xi_1^{(n)}}n\,, \tag{4.11}$$

$$Z_2^{(n)}(t) = \frac{\nu_1^{(n)}Y^{(n)}(t) - \xi_1^{(n)}X^{(n)}(t)}{\nu_1^{(n)}\xi_2^{(n)} - \nu_2^{(n)}\xi_1^{(n)}}n\,, \tag{4.12}$$

$$\begin{cases} E\left[\tilde{\gamma}_1^n\right] = \dfrac{\nu_1^{(n)}\eta_1^{(n)}}{\lambda_1^{(n)}}, & E\left[\tilde{\gamma}_2^n\right] = -\dfrac{\xi_1^{(n)}\eta_2^{(n)}}{\lambda_1^{(n)}}, \\ E\left[\tilde{\psi}_1^n\right] = \dfrac{\nu_2^{(n)}\eta_1^{(n)}}{\lambda_2^{(n)}}, & E\left[\tilde{\psi}_2^n\right] = -\dfrac{\xi_2^{(n)}\eta_2^{(n)}}{\lambda_2^{(n)}}, \end{cases} \tag{4.13}$$

we see by (4.13) that

$$\left\langle \frac{\lambda_1^{(n)}}{n}E\left[\tilde{\gamma}_2^n\right]g_2, Z_1^{(n)}(t)\right\rangle + \left\langle \frac{\lambda_2^{(n)}}{n}E\left[\tilde{\psi}_2^n\right]g_2, Z_2^{(n)}(t)\right\rangle = -\left\langle \eta_2^{(n)}g_2, Y^{(n)}(t)\right\rangle. \tag{4.14}$$

Hence the terms involving f_w cancel and (4.10) can be rewritten as

$$\begin{aligned} \tilde{h}^{(n)}(t) = \\ \frac{1}{2}\Bigg\langle & \frac{n\left(\xi_2^{(n)}F^{1(n)}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right) - \xi_1^{(n)}F^{2(n)}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\right)}{\nu_1^{(n)}\xi_2^{(n)} - \nu_2^{(n)}\xi_1^{(n)}} \\ & + 2f_x\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\eta_1^{(n)}g_1 \\ & + f_{xx}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)a_{11}^{(n)}(g_1)^2 \\ & + 2f_{xw}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)a_{12}^{(n)}g_1g_2 \\ & + f_{ww}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)a_{22}^{(n)}(g_2)^2, X^{(n)}(t)\Bigg\rangle \\ + \frac{1}{2}\Bigg\langle & \frac{n\left(\nu_2^{(n)}F^{1(n)}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right) - \nu_1^{(n)}F^{2(n)}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\right)}{\nu_1^{(n)}\xi_2^{(n)} - \nu_2^{(n)}\xi_1^{(n)}} \\ & + f_{xx}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)b_{11}^{(n)}(g_1)^2 \\ & + 2f_{xw}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)b_{12}^{(n)}g_1g_2 \\ & + f_{ww}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)b_{22}^{(n)}(g_2)^2, Y^{(n)}(t)\Bigg\rangle + O\left(n^{-1}|X^{(n)}(t)|\right), \end{aligned} \tag{4.15}$$

where

$$b_{ij}^{(n)} = \frac{-\lambda_1^{(n)}\nu_2^{(n)}a_{ij}^{1(n)} + \lambda_2^{(n)}\nu_1^{(n)}a_{ij}^{2(n)}}{n\left(\nu_1^{(n)}\xi_2^{(n)} - \nu_2^{(n)}\xi_1^{(n)}\right)}, \tag{4.16}$$

and $a_{ij}^{(n)}$, $a_{ij}^{k(n)}$ are defined in (3.16). Note that, by 3.8 and Lemma 3.1, the $b_{ij}^{(n)}$ are uniformly bounded in n. Finally, by (4.8) and (4.9), we obtain

$$\begin{aligned}\tilde{h}^{(n)}(t) = \frac{1}{2}\Big\langle & f_x\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\Delta g_1 \\ & + 2f_x\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\eta_1^{(n)}g_1 \\ & + f_{xx}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)a_{11}^{(n)}(g_1)^2 \\ & + 2f_{xw}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)a_{12}^{(n)}g_1g_2 \\ & + f_{ww}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)a_{22}^{(n)}(g_2)^2, X^{(n)}(t)\Big\rangle \\ + \frac{1}{2}\Big\langle & f_w\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\Delta g_2 \\ & + f_{xx}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)b_{11}^{(n)}(g_1)^2 \\ & + 2f_{xw}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)b_{12}^{(n)}g_1g_2 \\ & + f_{ww}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)b_{22}^{(n)}(g_2)^2, Y^{(n)}(t)\Big\rangle + O\left(n^{-1}|X^{(n)}(t)|\right).\end{aligned} \tag{4.17}$$

By (4.17) it is easy to see that

$$\tilde{h}^{(n)}(t) = O\left(|X^{(n)}(t)|\right). \tag{4.18}$$

Set

$$\begin{aligned} J^{(n)}(x,w)(\cdot) \equiv \tfrac{1}{2}\Big(& f_w(x,w)\Delta g_2(\cdot) + f_{xx}(x,w)\, b_{11}^{(n)}(g_1(\cdot))^2 \\ & + 2f_{xw}(x,w)\, b_{12}^{(n)}g_1(\cdot)g_2(\cdot) + f_{ww}(x,w)\, b_{22}^{(n)}g_2(\cdot)\Big), \end{aligned}$$

and

$$f^{(n)}(t) = \tilde{f}^{(n)}(t) + \left\langle \left(\eta_2^{(n)}\right)^{-1} J^{(n)}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right), Y^{(n)}(t)\right\rangle. \tag{4.19}$$

To find $h^{(n)}(t)$ so that $(f^{(n)}, h^{(n)}) \in \mathcal{A}_n$ we calculate $\lim_{\epsilon\to 0}\epsilon^{-1}E[f^{(n)}(t+\epsilon) - f^{(n)}(t)|\mathcal{F}_t^n]$ and obtain

$$\begin{aligned} h^{(n)}(t) = {} & \tilde{h}^{(n)}(t) + O\left(\left(\eta_2^{(n)}\right)^{-1}|X^{(n)}(t)|^2\right) + O\left(\left(\eta_2^{(n)}\right)^{-1}|X^{(n)}(t)|\right) \\ & - \left\langle J^{(n)}\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right), Y^{(n)}(t)\right\rangle. \end{aligned} \tag{4.20}$$

Combining terms we obtain the desired result. ∎

Conditions (1.6)— (1.8) now follow immediately from the following bound.

Lemma 4.2 *$E[\sup_{t\le T}|X^{(n)}(t)|^2]$ is uniformly bounded in n for each $T>0$.*

Proof From (4.7) we have that, for all $\epsilon>0$,

$$(4.21)\quad M_\epsilon^{(1)}(t) \equiv \exp\left\{-\epsilon\left|X^{(n)}(t)\right|^2\right\}$$
$$-\int_0^t \left\langle \lambda_1^{(n)} E_{\tilde\gamma^n}\left[\exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|+\frac{\tilde\gamma_1^n}{n}\right)^2\right\}-\exp\left\{-\epsilon\left|X^{(n)}(s)\right|^2\right\}\right], Z_1^{(n)}(s)\right\rangle$$
$$+\left\langle \lambda_2^{(n)} E_{\tilde\psi^n}\left[\exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|+\frac{\tilde\psi_1^n}{n}\right)^2\right\}-\exp\left\{-\epsilon\left|X^{(n)}(s)\right|^2\right\}\right], Z_2^{(n)}(s)\right\rangle ds$$

is a martingale. Hence

$$E\left[\epsilon\left|X^{(n)}(t)\right|^2\exp\left\{-\epsilon\left|X^{(n)}(t)\right|^2\right\}\right]$$
$$\le E\left[1-\exp\left\{-\epsilon\left|X^{(n)}(t)\right|^2\right\}\right]$$
$$= 1-E\left[\exp\left\{-\epsilon\left|X^{(n)}(0)\right|^2\right\}\right]$$
$$-E\left[\int_0^t \left\langle \lambda_1^{(n)} E_{\tilde\gamma^n}\left[\exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|+\frac{\tilde\gamma_1^n}{n}\right)^2\right\}-\exp\left\{-\epsilon\left|X^{(n)}(s)\right|^2\right\}\right], Z_1^{(n)}(s)\right\rangle\right.$$
$$\left.+\left\langle \lambda_2^{(n)} E_{\tilde\psi^n}\left[\exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|+\frac{\tilde\psi_1^n}{n}\right)^2\right\}-\exp\left\{-\epsilon\left|X^{(n)}(s)\right|^2\right\}\right], Z_2^{(n)}(s)\right\rangle ds\right]$$
$$= 1-E\left[\exp\left\{-\epsilon\left|X^{(n)}(0)\right|^2\right\}\right]$$
$$+E\left[\int_0^t \left\langle \lambda_1^{(n)}\exp\left\{-\epsilon\left|X^{(n)}(s)\right|^2\right\}\right.\right.$$
$$\left.\times E_{\tilde\gamma^n}\left[1-\exp\left\{-\epsilon\left(\left(\left|X^{(n)}(s)\right|+\frac{\tilde\gamma_1^n}{n}\right)^2-\left|X^{(n)}(s)\right|^2\right)\right\}\right], Z_1^{(n)}(s)\right\rangle$$
$$+\left\langle \lambda_2^{(n)}\exp\left\{-\epsilon\left|X^{(n)}(s)\right|^2\right\}\right.$$
$$\left.\left.\times E_{\tilde\psi^n}\left[1-\exp\left\{-\epsilon\left(\left(\left|X^{(n)}(s)\right|+\frac{\tilde\psi_1^n}{n}\right)^2-\left|X^{(n)}(s)\right|^2\right)\right\}\right], Z_2^{(n)}(s)\right\rangle ds\right]$$
$$\le 1-E\left[\exp\left\{-\epsilon\left|X^{(n)}(0)\right|^2\right\}\right]$$

$$+E\left[\int_0^t \left\langle \lambda_1^{(n)} \exp\left\{-\epsilon \left|X^{(n)}(s)\right|^2\right\} \epsilon E_{\tilde{\gamma}^n}\left[\left(\left|X^{(n)}(s)\right| + \frac{\tilde{\gamma}_1^n}{n}\right)^2 - \left|X^{(n)}(s)\right|^2\right], Z_1^{(n)}(s)\right\rangle\right.$$

$$\left. + \left\langle \lambda_2^{(n)} \exp\left\{-\epsilon \left|X^{(n)}(s)\right|^2\right\} \epsilon E_{\tilde{\psi}^n}\left[\left(\left|X^{(n)}(s)\right| + \frac{\tilde{\psi}_1^n}{n}\right)^2 - \left|X^{(n)}(s)\right|^2\right], Z_2^{(n)}(s)\right\rangle ds\right]$$

$$\leq \epsilon E\left[\left|X^{(n)}(0)\right|^2\right]$$

$$+ \epsilon E\left[\int_0^t \exp\left\{-\epsilon \left|X^{(n)}(s)\right|^2\right\} \left(\frac{\lambda_1^{(n)}}{n} E_{\tilde{\gamma}^n}\left[(\tilde{\gamma}_1^n)^2\right] \left|\frac{Z_1^{(n)}(s)}{n}\right|\right.\right.$$

$$+ \frac{\lambda_2^{(n)}}{n} E_{\tilde{\psi}^n}\left[\left(\tilde{\psi}_1^n\right)^2\right] \left|\frac{Z_2^{(n)}(s)}{n}\right| + 2\frac{\lambda_1^{(n)}}{n} E_{\tilde{\gamma}^n}\left[\tilde{\gamma}_1^n\right] \left|X^{(n)}(s)\right| \left|Z_1^{(n)}(s)\right|$$

$$\left.\left. + 2\frac{\lambda_2^{(n)}}{n} E_{\tilde{\psi}^n}\left[\tilde{\psi}_1^n\right] \left|X^{(n)}(s)\right| \left|Z_2^{(n)}(s)\right|\right) ds\right].$$

By the uniform boundedness of $\lambda_i^{(n)}/n$, $E_{\tilde{\gamma}^n}[(\tilde{\gamma}_1^n)^2]$, $E_{\tilde{\psi}^n}[(\tilde{\psi}_1^n)^2]$, and the definition of $X^{(n)}$, there exists a constant C_1, independent of n, such that

$$E\left[\epsilon \left|X^{(n)}(t)\right|^2 \exp\left\{-\epsilon \left|X^{(n)}(t)\right|^2\right\}\right] \tag{4.22}$$

$$\leq \epsilon E\left[\left|X^{(n)}(0)\right|^2\right] + \epsilon C_1 \int_0^t E\left[\left|X^{(n)}(s)\right|\right] ds$$

$$+ 2\epsilon\eta_1^{(n)} \int_0^t E\left[\exp\left\{-\epsilon \left|X^{(n)}(s)\right|^2\right\} \left|X^{(n)}(s)\right|^2\right] ds.$$

Recalling that $|X^{(n)}(t)| \exp\{-\eta_1^{(n)} t\}$ is a martingale (c.f. [1]), (4.22) can be rewritten as:

$$E\left[\epsilon \left|X^{(n)}(t)\right|^2 \exp\left\{-\epsilon \left|X^{(n)}(t)\right|^2\right\}\right] \tag{4.23}$$

$$\leq \epsilon E\left[\left|X^{(n)}(0)\right|^2\right] + \epsilon C_1 \int_0^t \exp\left\{\eta_1^{(n)} s\right\} E\left[\left|X^{(n)}(0)\right|\right] ds$$

$$+ 2\epsilon\eta_1^{(n)} \int_0^t E\left[\exp\left\{-\epsilon \left|X^{(n)}(s)\right|^2\right\} \left|X^{(n)}(s)\right|^2\right] ds$$

$$\leq \epsilon E\left[\left|X^{(n)}(0)\right|^2\right] + \epsilon C_1 t \exp\left\{\eta_1^{(n)} t\right\} E\left[\left|X^{(n)}(0)\right|\right]$$

$$+ 2\epsilon\eta_1^{(n)} \int_0^t E\left[\exp\left\{-\epsilon \left|X^{(n)}(s)\right|^2\right\} \left|X^{(n)}(s)\right|^2\right] ds.$$

By (4.23) and Gronwall's inequality we obtain

$$E\left[\epsilon \left|X^{(n)}(t)\right|^2 \exp\left\{-\epsilon \left|X^{(n)}(t)\right|^2\right\}\right] \tag{4.24}$$

$$\leq \epsilon \left(E\left[\left|X^{(n)}(0)\right|^2\right] + C_1 t \exp\left\{\eta_1^{(n)} t\right\} E\left[\left|X^{(n)}(0)\right|\right]\right) \exp\left\{2\eta_1^{(n)} t\right\}$$

Dividing by ϵ and letting $\epsilon \downarrow 0$ gives

$$(4.25) \qquad E\left[\left|X^{(n)}(t)\right|^2\right] \le \left(E\left[\left|X^{(n)}(0)\right|^2\right] + C_1 t \exp\left\{\eta_1^{(n)} t\right\} E\left[\left|X^{(n)}(0)\right|\right]\right) \exp\left\{2\eta_1^{(n)} t\right\}.$$

That is, $E[|X^{(n)}(t)|^2]$ is uniformly bounded in n. Once again using the martingale properties of $|X^{(n)}(t)|$, we obtain, by Doob's inequality, that $E[\sup_{t \le T} |X^{(n)}(t)|^2]$ is also uniformly bounded in n for each $T > 0$. ∎

4.2 Compact containment

Definition 4.3 *Let (E, τ) be a completely regular topological space with metrizable compacts. Suppose that $X^{(n)}$, $n = 1, 2, \ldots$, is a $\mathcal{F}_t^n$-adapted process with sample paths in $D_E[0, \infty)$. We say that the compact containment condition holds for $\{X^{(n)}\}$, if, for every $\epsilon > 0$ and $T > 0$, there exists a compact set $\Gamma_{\epsilon,T} \subset E$ for which*

$$\inf_n P\left\{X^{(n)}(t) \in \Gamma_{\epsilon,T} \quad \text{for all} \quad 0 \le t \le T\right\} \ge 1 - \epsilon.$$

In the following lemmas we shall denote $\sum_{i=1}^d (\partial_i g)^2$ by $(\partial g)^2$.

Lemma 4.4 *$E[\langle g_2, W^{(n)}(t)\rangle^2]$ is bounded uniformly in n for each $g_2 \in S(R^d)$.*

Proof It is easy to see that for each $\epsilon > 0$

$$\exp\left\{-\epsilon\left(|\mu_1|^2 + \langle g_2, \mu_2\rangle^2\right)\right\} \in \mathcal{D}(A_1).$$

From (4.7) we obtain that

$$\begin{aligned}
M_\epsilon^{(2)}(t) \equiv{} & \exp\left\{-\epsilon\left(\left|X^{(n)}(t)\right|^2 + \left\langle g_2, W^{(n)}(t)\right\rangle^2\right)\right\} \\
(4.26) \qquad & - \int_0^t \Bigg\langle \epsilon \exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|^2 + \left\langle g_2, W^{(n)}(s)\right\rangle^2\right)\right\} \\
& \qquad \times \left[\left(2\epsilon\left\langle g_2, W^{(n)}(s)\right\rangle^2 - 1\right)\left(\frac{\xi_1^{(n)}}{n}\right)^2 (\partial g_2)^2 - \left\langle g_2, W^{(n)}(s)\right\rangle \frac{\xi_1^{(n)}}{n} \Delta g_2\right] \\
& \qquad + \lambda_1^{(n)} E_{\tilde{\gamma}^n}\Bigg[\exp\left\{-\epsilon\left(\left(\left|X^{(n)}(s)\right| + \frac{\tilde{\gamma}_1^n}{n}\right)^2 + \left(\left\langle g_2, W^{(n)}(s)\right\rangle + \frac{\tilde{\gamma}_2^n}{n} g_2\right)^2\right)\right\} \\
& \qquad - \exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|^2 + \left\langle g_2, W^{(n)}(t)\right\rangle^2\right)\right\}\Bigg], Z_1^{(n)}(s)\Bigg\rangle \\
& + \Bigg\langle \epsilon \exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|^2 + \left\langle g_2, W^{(n)}(s)\right\rangle^2\right)\right\} \\
& \qquad \times \left[\left(2\epsilon\left\langle g_2, W^{(n)}(s)\right\rangle^2 - 1\right)\left(\frac{\xi_2^{(n)}}{n}\right)^2 (\partial g_2)^2 - \left\langle g_2, W^{(n)}(s)\right\rangle \frac{\xi_2^{(n)}}{n} \Delta g_2\right]
\end{aligned}$$

$$+\lambda_2^{(n)}E_{\tilde{\psi}^n}\left[\exp\left\{-\epsilon\left(\left(\left|X^{(n)}(s)\right|+\frac{\tilde{\psi}_1^n}{n}\right)^2+\left(\left\langle g_2,W^{(n)}(s)\right\rangle+\frac{\tilde{\psi}_2^n}{n}g_2\right)^2\right)\right\}\right.$$

$$\left.-\exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|^2+\left\langle g_2,W^{(n)}(t)\right\rangle^2\right)\right\}\right],Z_2^{(n)}(s)\Big\rangle$$

$$-2\epsilon\left\langle g_2,W^{(n)}(s)\right\rangle\exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|^2+\left\langle g_2,W^{(n)}(s)\right\rangle^2\right)\right\}\eta_2^{(n)}\left\langle g_2,Y^{(n)}(s)\right\rangle ds$$

is a martingale. By simple calculations (as in the previous lemma) we obtain that

$$E\left[\epsilon\left\langle g_2,W^{(n)}(t)\right\rangle^2\exp\left\{-\epsilon\left(\left|X^{(n)}(t)\right|^2+\left\langle g_2,W^{(n)}(t)\right\rangle^2\right)\right\}\right]$$

$$\leq \epsilon E\left[O\left(\left|X^{(n)}(0)\right|^2\right)\right]+\epsilon\int_0^t E\left[O\left(\left|X^{(n)}(s)\right|^2\right)\right]ds+\epsilon\int_0^t E\left[O\left(\left|X^{(n)}(s)\right|\right)\right]ds$$

$$+\tfrac{1}{2}\int_0^t E\left[\epsilon\exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|^2+\left\langle g_2,W^{(n)}(s)\right\rangle^2\right)\right\}\left\langle g_2,W^{(n)}(s)\right\rangle^2\right]ds$$

$$\equiv \epsilon B_n(t)+\tfrac{1}{2}\int_0^t E\left[\epsilon\exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right|^2+\left\langle g_2,W^{(n)}(s)\right\rangle^2\right)\right\}\left\langle g_2,W^{(n)}(s)\right\rangle^2\right]ds,$$

where the $B_n(t)\geq 0$ are bounded uniformly in n. (This follows from the uniform boundedness of $E[\sup_{s\leq t}|X^{(n)}(s)|^2]$). Hence there exists a function B such that $B(t)\geq B_n(t)$ for each n. By Gronwall's inequality we obtain

$$E\left[\left\langle g_2,W^{(n)}(t)\right\rangle^2\exp\left\{-\epsilon\left(\left|X^{(n)}(t)\right|^2+\left\langle g_2,W^{(n)}(t)\right\rangle^2\right)\right\}\right]\leq B_n(t)\exp\left\{\tfrac{1}{2}t\right\},\quad\forall\epsilon>0.$$

Letting $\epsilon\downarrow 0$ we find, by monotone convergence,

$$E\left[\left\langle g_2,W^{(n)}(t)\right\rangle^2\right]\leq B_n(t)\exp\left\{\tfrac{1}{2}t\right\}\leq B(t)\exp\left\{\tfrac{1}{2}t\right\},\tag{4.27}$$

so that $E[\langle g_2,W^{(n)}(t)\rangle^2]$ is uniformly bounded in n.

■

Lemma 4.5

$$\left\langle g_2,W^{(n)}(t)\right\rangle-\tfrac{1}{2}\int_0^t\left\langle\Delta g_2,Y^{(n)}(s)\right\rangle ds\tag{4.28}$$

is a martingale for each n and $g_2\in S(R^d)$.

Proof Note that $\exp\{-\epsilon|\mu_1|\}\sin(\epsilon\langle g_2,\mu_2\rangle)\in\mathcal{D}(A_1)$. Define a martingale $M_\epsilon^{(3)}(t)$ as in the previous cases :

$$M_\epsilon^{(3)}(t)\equiv\exp\left(-\epsilon\left|X^{(n)}(t)\right|\right)\sin\left(\epsilon\left\langle g_2,W^{(n)}(t)\right\rangle\right)$$

(4.29)

$$-\int_0^t\left\langle\epsilon\exp\left\{-\epsilon\left|X^{(n)}(s)\right|\right\}\left(\cos\left(\epsilon\left\langle g_2,W^{(n)}(s)\right\rangle\right)\frac{\xi_1^{(n)}}{n}\Delta g_2\right.\right.$$

$$
\begin{aligned}
&- \epsilon \sin\left(\epsilon\left\langle g_2, W^{(n)}(s)\right\rangle\right)\left(\frac{\xi_1^{(n)}}{n}\right)^2 (\partial g_2)^2\Big), Z_1^{(n)}(t)\Big\rangle \\
&+ \Big\langle \epsilon \exp\left\{-\epsilon\left|X^{(n)}(s)\right|\right\}\Big(\cos\left(\epsilon\left\langle g_2, W^{(n)}(s)\right\rangle\right)\frac{\xi_2^{(n)}}{n}\Delta g_2 \\
&- \epsilon \sin\left(\epsilon\left\langle g_2, W^{(n)}(s)\right\rangle\right)\left(\frac{\xi_2^{(n)}}{n}\right)^2 (\partial g_2)^2\Big), Z_2^{(n)}(t)\Big\rangle \\
&+ \Big\langle \lambda_1^{(n)} E_{\tilde{\gamma}^n}\Big[\exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right| + \frac{\tilde{\gamma}_1^n}{n}\right)\right\}\sin\left(\epsilon\left\langle g_2, W^{(n)}(s)\right\rangle + \frac{\tilde{\gamma}_2^n}{n} g_2\right) \\
&- \exp\left\{-\epsilon\left|X^{(n)}(s)\right|\right\}\sin\left(\epsilon\left\langle g_2, W^{(n)}(s)\right\rangle\right)\Big], Z_1^{(n)}(s)\Big\rangle \\
&+ \Big\langle \lambda_2^{(n)} E_{\tilde{\psi}^n}\Big[\exp\left\{-\epsilon\left(\left|X^{(n)}(s)\right| + \frac{\tilde{\psi}_1^n}{n}\right)\right\}\sin\left(\epsilon\left\langle g_2, W^{(n)}(s)\right\rangle + \frac{\tilde{\psi}_2^n}{n} g_2\right) \\
&- \exp\left\{-\epsilon\left|X^{(n)}(s)\right|\right\}\sin\left(\epsilon\left\langle g_2, W^{(n)}(s)\right\rangle\right)\Big], Z_2^{(n)}(s)\Big\rangle \\
&+ \epsilon \exp\left\{-\epsilon\left|X^{(n)}(s)\right|\right\}\cos\left(\epsilon\left\langle g_2, W^{(n)}(s)\right\rangle\right)\eta_2^{(n)}\left\langle g_2, Y^{(n)}(s)\right\rangle ds
\end{aligned}
$$

Let

(4.30)
$$
\begin{aligned}
M^{(3)}(t) &\equiv \lim_{\epsilon\to 0}\frac{M_\epsilon^{(3)}(t)}{\epsilon} \\
&= \left\langle g_2, W^{(n)}(t)\right\rangle - \int_0^t \left\langle \lambda_1^{(n)} E_{\tilde{\gamma}^n}\left[\frac{\tilde{\gamma}_2^n}{n}\right] g_2, Z_1^{(n)}(s)\right\rangle \\
&\quad + \left\langle \lambda_2^{(n)} E_{\tilde{\psi}^n}\left[\frac{\tilde{\psi}_2^n}{n}\right] g_2, Z_2^{(n)}(s)\right\rangle + \tfrac{1}{2}\left\langle \Delta g_2, Y^{(n)}(s)\right\rangle \\
&\quad + \eta_2^{(n)}\left\langle g_2, Y^{(n)}(s)\right\rangle ds \\
&= \left\langle g_2, W^{(n)}(t)\right\rangle - \tfrac{1}{2}\int_0^t \left\langle \Delta g_2, Y^{(n)}(s)\right\rangle ds,
\end{aligned}
$$

where the last equality follows by definition of $Y^{(n)}$. The above limit exists almost surely. On the other hand, it is easy to see from (4.29) that

$$
\left|\frac{M_\epsilon^{(3)}(t)}{\epsilon}\right| \le \left|\left\langle g_2, W^{(n)}(t)\right\rangle\right| + \int_0^t C_1\left|X^{(n)}(s)\right|\left|\left\langle g_2, W^{(n)}(s)\right\rangle\right| + C_2\left|X^{(n)}(s)\right| ds
$$

where $C_1 > 0$ and $C_2 > 0$ do not depend on ϵ. From the uniform boundedness of $E[|X^{(n)}(t)|^2]$ and $E[\langle g_2, W^{(n)}(t)\rangle^2]$, we obtain the uniform integrability of $|M_\epsilon^{(3)}(t)/\epsilon|$, which means that convergence is in L_1. Consequently $M^{(3)}(t)$ is a martingale. ∎

Lemma 4.6 *$E[\sup_{t\le T}\langle g_2, W^{(n)}(t)\rangle^2]$ is bounded uniformly in n for each $g_2 \in S(R^d)$.*

Proof From the previous lemma we have that

$$\left\langle g_2, W^{(n)}(t)\right\rangle = M^{(3)}(t) + \tfrac{1}{2}\int_0^t \left\langle \Delta g_2, Y^{(n)}(s)\right\rangle ds.$$

Then

$$\begin{aligned}
E\left[\sup_{t\le T}\left\langle g_2, W^{(n)}(t)\right\rangle^2\right] &= E\left[\sup_{t\le T}\left(M^{(3)}(t) + \tfrac{1}{2}\int_0^t \left\langle \Delta g_2, Y^{(n)}(s)\right\rangle ds\right)^2\right] \\
&\le 2E\left[\sup_{t\le T}\left(M^{(3)}(t)\right)^2\right] + 2E\left[\sup_{t\le T}\tfrac{1}{4}\left(\int_0^t \left\langle \Delta g_2, Y^{(n)}(s)\right\rangle ds\right)^2\right] \\
&\le 8E\left[\left(M^{(3)}(T)\right)^2\right] + \tfrac{1}{2}\left\|\Delta g_2\right\|^2 T^2 C E\left[\sup_{t\le T}\left|X^{(n)}(t)\right|^2\right] \\
&\le 8E\left[2\left\langle g_2, W^{(n)}(T)\right\rangle^2 + \tfrac{1}{2}\left(\int_0^T \left\langle \Delta g_2, Y^{(n)}(s)\right\rangle ds\right)^2\right] \\
&\quad + \tfrac{1}{2}\left\|\Delta g_2\right\|^2 T^2 C E\left[\sup_{t\le T}\left|X^{(n)}(t)\right|^2\right],
\end{aligned}$$

where C is the constant such that

$$\left|\left\langle \Delta g_2, Y^{(n)}(t)\right\rangle\right|^2 \le C\left\|\Delta g_2\right\|^2 \left|X^{(n)}(t)\right|^2, \ \forall n.$$

Finally we obtain that

$$E\left[\sup_{t\le T}\left\langle g_2, W^{(n)}(t)\right\rangle^2\right] \le 16E\left[\left\langle g_2, W^{(n)}(T)\right\rangle^2\right] + \left\|\Delta g_2\right\|^2 T^2 C E\left[\sup_{t\le T}\left|X^{(n)}(t)\right|^2\right].$$

From Lemmas 4.2 and 4.4 we obtain uniform in n boundedness of $E[\sup_{t\le T}\left\langle g_2, W^{(n)}(t)\right\rangle^2]$. ∎

Lemma 4.7 *For each $g_1 \in \mathcal{D}(\frac{1}{2}\Delta)_+$, $g_2 \in S(R^d)$, the processes $\langle g_1, X^{(n)}(t)\rangle$, $\langle g_2, W^{(n)}(t)\rangle$ and $(\langle g_1, X^{(n)}(t)\rangle, \langle g_2, W^{(n)}(t)\rangle)$ satisfy the compact containment condition.*

Proof The proof is immediate by the uniform boundedness of

$$E\left[\sup_{t\le T}\left\langle g_2, W^{(n)}(t)\right\rangle^2\right], \quad E\left[\sup_{t\le T}\left\langle g_1, X^{(n)}(t)\right\rangle^2\right]$$

and Chebyshev's inequality in the case of the processes $\langle g_1, X^{(n)}(t)\rangle$, $\langle g_2, W^{(n)}(t)\rangle$. In the case of the process $(\langle g_1, X^{(n)}(t)\rangle, \langle g_2, W^{(n)}(t)\rangle)$ the result follows from the fact that the product of two compact sets is compact in the product space. ∎

4.3 Relative compactness

We now check the relative compactness of $\{(X^{(n)}, W^{(n)})\}$. *En passant*, we shall show that $X^{(n)}$ converges weakly to super-Brownian motion; i.e. the process with values in M_F which is the unique solution of the martingale problem for $\mathbf{A}^*$ with

$$\mathbf{A}^* = \left\{\exp\{-\langle g_1,\mu\rangle\},\ \exp\{-\langle g_1,\mu\rangle\}\left\langle -\eta_1 g_1 + \tfrac{1}{2}a_{11}(g_1)^2 - \tfrac{1}{2}\Delta g_1, \mu\right\rangle : g_1 \in \tilde{\mathcal{D}}\left(\tfrac{1}{2}\Delta\right)_+\right\}.$$

Recall that $\tilde{\mathcal{D}}(\frac{1}{2}\Delta)_+ \equiv \mathcal{D}(\frac{1}{2}\Delta) \cap C_l(R^d)_+$.

Lemma 4.8 *The following sequences of processes are tight for each $g_1 \in \mathcal{D}(\frac{1}{2}\Delta)_+$, $g_2 \in S(R^d)$:*

1) $\left\{\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\right\}$ *in* $D_{R^2}[0,\infty)$,

2) $\left\{\left\langle g_1, X^{(n)}(t)\right\rangle\right\}$, $\left\{\left\langle g_2, W^{(n)}(t)\right\rangle\right\}$ *in* $D_R[0,\infty)$,

3) $\left\{\left\langle g_1, X^{(n)}(t)\right\rangle + \left\langle g_2, W^{(n)}(t)\right\rangle\right\}$ *in* $D_R[0,\infty)$.

Proof 2 and 3 are simple corollaries of 1, which we now prove. By Lemma 4.1 we obtain that, for all $G_{f,g_1,g_2} \in \mathcal{D}(A_1)$, we can choose the processes $f^{(n)}$ and $h^{(n)}$, $(f^{(n)}, h^{(n)}) \in \mathcal{A}_n$ such that

$$\begin{aligned}\lim_{n\uparrow\infty} E\left[\sup_{t\le T}\left|f^{(n)}(t) - f\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\right|\right] &= \lim_{n\uparrow\infty} E\left[\sup_{t\le T} O\left(\left(\eta_2^{(n)}\right)^{-1}|X^{(n)}(t)|\right)\right] \\ &= 0.\end{aligned}$$

The last line follows from the uniform boundedness of $E[\sup_{t\le T}|X^{(n)}(t)|]$. Furthermore,

$$\begin{aligned}\sup_n E\left[\sup_{t\le T}\left|h^{(n)}(t)\right|\right] &\le \sup_n E\left[\sup_{t\le T}\left|A_1 f\left(\left\langle g_1, X^{(n)}(t)\right\rangle, \left\langle g_2, W^{(n)}(t)\right\rangle\right)\right|\right] \\ &\quad + \sup_n E\left[\sup_{t\le T} O\left(n^{-1}\left|X^{(n)}(t)\right|\right)\right] \\ &\quad + \sup_n E\left[\sup_{t\le T} O\left(\left(\eta_2^{(n)}\right)^{-1}|X^{(n)}(t)|\right)\right] \\ &\quad + \sup_n E\left[\sup_{t\le T} O\left(\left(\eta_2^{(n)}\right)^{-1}|X^{(n)}(t)|^2\right)\right] + \sup_n o(1) \\ &< \infty,\end{aligned}$$

where the final line follows from the uniform boundedness of $E[\sup_{t\le T}|X^{(n)}(t)|^2]$, and the boundedness of functions in the range of the operator A_1. Since this holds for each choice of g_1, g_2, and by Lemma 4.7 we have compact containment, applying Theorems 3.9.1, 3.9.4 [5] now completes the proof. ∎

Lemma 4.8 and Mitoma's theorem [10] immediately yield

Lemma 4.9 *$\{W^{(n)}\}$ is a tight sequence of processes in $D_{S'}[0,\infty)$.*

Lemma 4.10 *$\{X^{(n)}\}$ converges weakly to super-Brownian motion.*

Proof We can take $g_2 = 0$ and thus obtain $\mathbf{A} \subset \overline{C}(M_F) \times \overline{C}(M_F)$, where $\mathbf{A} \subset A_1$ is defined as follows:

$$\mathbf{D}(\mathbf{A}) = \text{linear span of } \left\{ \exp\left\{-\langle g_1, \mu\rangle\right\},\ g_1 \in \tilde{\mathcal{D}}\left(\tfrac{1}{2}\Delta\right)_+ \right\}$$

$$\mathbf{A}\exp\left\{-\langle g_1, \mu\rangle\right\} = \exp\left\{-\langle g_1, \mu\rangle\right\} \left\langle -\eta_1 g_1 + \frac{1}{2}a_{11}(g_1)^2 - \frac{1}{2}\Delta g_1, \mu \right\rangle.$$

By Theorem 2.4 [14] and results of Perkins [12] the closure of $\mathbf{A}$ generates a strongly continuous contraction semigroup on $\overline{C}(M_F)$ (with corresponding super-Brownian motion X) and $\mathbf{D}(\mathbf{A})$ is a core for $\overline{\mathbf{A}}$. By Theorem 3.2.6 [4] the set of functions $\{\exp\{-\langle g_1, \mu\rangle\}\}$ strongly separates points in M_F. Thus, all the assumptions of Theorem 4.8.2 [5] are satisfied and we are done. ∎

Lemma 4.11 $\{(X^{(n)}, W^{(n)})\}$ *is a relatively compact sequence of processes.*

Proof We check the conditions of Theorem 4.6 [7] (Jakubowski's tightness criterion). $\{X^{(n)}\}$ and $\{W^{(n)}\}$ are tight. This implies that the compact containment condition holds for each of these processes and by the same arguments as in Lemma 4.7 it holds for the pair $\{(X^{(n)}, W^{(n)})\}$. Define the family of functions $F : M_F \times S' \longrightarrow R$ by

$$F_{g_1, g_2}(\mu_1, \mu_2) = \langle g_1, \mu_1\rangle + \langle g_2, \mu_2\rangle, \quad g_1 \in \mathcal{D}(\tfrac{1}{2}\Delta)_+,\ g_2 \in S(R^d).$$

This family separates points in $M_F \times S'$ and is closed under addition. By Lemma 4.8 the sequence $\{f(X^{(n)}, W^{(n)})\}$ is tight for all $f \in F$. Jakubowski's conditions are satisfied and the proof is complete. ∎

Theorem 1.3 and Lemmas 4.11, 4.1 now imply

Corollary 4.12 *Each limit point of* $\{(X^{(n)}, W^{(n)})\}$ *is a solution of the* $D_{M_F \times S'}[0,\infty)$ *martingale problem for* (A_1, ν).

4.4 Continuity of the limit process

Throughout this subsection we shall denote by W one of the limit points of the sequence $\{W^{(n)}\}$, and all results are obtained for each limit point of $\{W^{(n)}\}$. For simplicity of notation we shall denote $\langle g, W(t)\rangle$ and $\langle g, X(t)\rangle$ by $W_t(g)$ and $X_t(g)$ respectively, while the increasing process of each martingale M_t will be $\langle M\rangle_t$.

We shall treat only the continuity of the process W_t, since the continuity of X follows from the fact that X is super Brownian motion. Using this result we shall prove the last part of Theorem 3.3, viz. that $\langle g_2, Y^{(n)}(t) - Y^{(n)}(0)\exp\{-\eta_2^{(n)}t\}\rangle$ converges in probability to zero for each $g_2 \in S(R^d)$.

Lemma 4.13 *For each* $g_2 \in S(R^d)$, $W_t(g_2)$ *is a square integrable martingale with increasing process*

$$\langle W(g_2)\rangle_t = \int_0^t a_{22} X_s\left(g_2^2\right) ds. \tag{4.31}$$

Proof The uniform boundedness in n of $E[(W_t^{(n)}(g_2))^2]$ (obtained during the proof of the compact containment condition) immediately gives the existence of second moments of $W_t(g_2)$. We now use the same "trick" as in Lemma 4.5.

$$
\begin{aligned}
(4.32) \qquad M_\epsilon^{(1)}(t) \equiv\ & \exp(-\epsilon|X_t|)\sin(\epsilon W_t(g_2)) \\
& - \int_0^t \epsilon^2 \exp\{-\epsilon|X_s|\}\Big(\sin(\epsilon W_s(g_2))\frac{a_{11}}{2}|X_s| \\
& - \cos(\epsilon W_s(g_2))a_{12}X_s(g_2) - \sin(\epsilon W_s(g_2))\frac{a_{22}}{2}X_s\left(g_2^2\right)\Big) \\
& - \epsilon\exp\{-\epsilon|X_s|\}\sin(\epsilon W_s(g_2))\eta_1|X_s|\, ds
\end{aligned}
$$

is a martingale. Since

$$
(4.33) \qquad \lim_{\epsilon\to 0}\frac{M_\epsilon^{(1)}(t)}{\epsilon} = W_t(g_2) \quad a.s.,
$$

the dominated convergence theorem gives that $W_t(g_2)$ is a martingale. Now note that

$$
\begin{aligned}
(4.34) \quad M_\epsilon^{(2)}(t) \equiv\ & \exp\{-\epsilon|X_t|)\sin\left(\epsilon W_t(g_2)^2\right\} \\
& - \int_0^t \epsilon\exp\{-\epsilon|X_s|\}\Big(\epsilon\sin\left(\epsilon W_s(g_2)^2\right)\frac{a_{11}}{2}|X_s| \\
& - 2\epsilon\cos\left(\epsilon W_s(g_2)^2\right)W_s(g_2)\,a_{12}X_s(g_2) + \cos\left(\epsilon W_s(g_2)^2\right)a_{22}X_s\left(g_2^2\right) \\
& - 2\epsilon W_s(g_2)^2\sin\left(\epsilon W_s(g_2)^2\right)a_{22}X_s\left(g_2^2\right) \\
& - \sin\left(\epsilon W_s(g_2)^2\right)\eta_1|X_s|\Big)\, ds
\end{aligned}
$$

is a martingale. Under the same arguments as in (4.33)

$$
(4.35) \qquad W_t(g_2)^2 - \int_0^t a_{22}X_s\left(g_2^2\right) ds = \lim_{\epsilon\to 0}\frac{M_\epsilon^{(2)}(t)}{\epsilon}, \quad a.s.
$$

is a martingale and so $\langle W(g_2)\rangle_t = \int_0^t a_{22}X_s(g_2^2)\, ds$. ∎

Lemma 4.14 *Assume that $W_0 \in S'$. Then, for each $g_2 \in S(R^d)$, $W_t(g_2)$ is a continuous martingale.*

Proof Define

$$
(4.36) \qquad \widetilde{W}_t^K(g_2) \equiv W_t(g_2) - K, \text{ for all } K \in R.
$$

Once again we use the same "trick".

$$M_\epsilon^{(3)}(t) \equiv \exp\left\{-\epsilon\,|X_t| - \left(\widetilde{W}_t^K(g_2)^2 - \widetilde{W}_0^K(g_2)^2\right)\right\}$$

$$- \int_0^t \epsilon \exp\left\{-\epsilon\,|X_s| - \left(\widetilde{W}_s^K(g_2)^2 - \widetilde{W}_0^K(g_2)^2\right)\right\}\left(\epsilon^2 \frac{a_{11}}{2}\,|X_s| + 2\epsilon \widetilde{W}_s^K(g_2)^2\, a_{12} X_s(g_2)\right.$$

$$\left. + \left(2\widetilde{W}_s^K(g_2)^2 - 1\right) a_{22} X_s\left(g_2^2\right) - \epsilon\eta_1\,|X_s|\right) ds$$

is a martingale. Under the same arguments as in (4.33) we obtain that

$$\begin{aligned} M^{(3)}(t) &= \exp\left\{-\left(\widetilde{W}_t^K(g_2)^2 - \widetilde{W}_0^K(g_2)^2\right)\right\} \\ &\quad - \int_0^t \exp\left\{-\left(\widetilde{W}_s^K(g_2)^2 - \widetilde{W}_0^K(g_2)^2\right)\right\}\left(2\widetilde{W}_s^K(g_2)^2 - 1\right) a_{22} X_s\left(g_2^2\right) ds \\ &= \lim_{\epsilon\to 0} M_\epsilon^{(3)}(t),\ a.s. \end{aligned}$$

is a martingale. By Corollary 2.3.3 from [5] we have that the process

$$M^{(4)}(t) \equiv \exp\left\{-\left(\widetilde{W}_t^K(g_2)^2 - \widetilde{W}_0^K(g_2)^2\right) - \int_0^t \left(2\widetilde{W}_s^K(g_2)^2 - 1\right) a_{22} X_s\left(g_2^2\right) ds\right\}$$

$$= \exp\left\{-\left(\widetilde{W}_t^K(g_2)^2 - \widetilde{W}_0^K(g_2)^2\right) + \int_0^t a_{22} X_s\left(g_2^2\right) ds - 2\int_0^t a_{22}\widetilde{W}_s^K(g_2)^2\, X_s\left(g_2^2\right) ds\right\}$$

is, at least, local martingale. By setting $\widetilde{g}_2 = m^2 g_2$ in the above, we see that, for every $m \in R$,

$$\exp\left\{-m^2\left[\left(\widetilde{W}_t^K(g_2)^2 - \widetilde{W}_0^K(g_2)^2\right) - \int_0^t a_{22} X_s\left(g_2^2\right) ds\right]\right. \\ \left. - \frac{m^4}{2}\int_0^t 4a_{22}\widetilde{W}_s^K(g_2)^2\, X_s\left(g_2^2\right) ds\right\} \tag{4.37}$$

is a local martingale. By Lemma 4.13

$$\widetilde{W}_t^K(g_2)^2 - \widetilde{W}_0^K(g_2)^2 - \int_0^t a_{22} X_s\left(g_2^2\right) ds$$

is a martingale. The continuity of this martingale derives from the result given in the following lemma [13]:

Lemma 4.15 *Let X be a local martingale such that $X_0 = 0$ and A be a continuous increasing process such that $A_0 = 0$. If $\exp\{kX_t - (k^2/2)A_t\}$ is a local martingale for every $k \in R_+$, then X is continuous and $A_t = \langle X\rangle_t$.*

Thus, by (4.37), we have that $\widetilde{W}_t^K(g_2)^2 - \widetilde{W}_0^K(g_2)^2$ is a continuous submartingale. This is true for every $K \in R$ and, thus,

$$\begin{aligned} \widetilde{W}_t^K(g_2)^2 - \widetilde{W}_t^0(g_2)^2 &= (W_t(g_2) - K)^2 - W_t(g_2)^2 \\ &= K^2 - 2KW_t(g_2) \end{aligned}$$

is a continuous process for each K, which implies that $W_t(g_2)$ is continuous. ∎

Lemma 4.14 and Theorem 1 [9] immediately yield

Corollary 4.16 *W_t is a continuous process taking values in S'.*

Corollary 4.17 *For $T > 0$, $\sup_{t \le T} |\langle g_2, Y^{(n)}(t)\rangle - \langle g_2, Y^{(n)}(0)\rangle \exp\{-\eta_2^{(n)} t\}|$ converges to zero in probability for each $g_2 \in S(R^d)$.*

Proof The proof is completely analogous to that of Theorem 9.2.1(b) [5], using the fact that, for each $g_2 \in S(R^d)$, $W_t(g_2)$ is a continuous process. ∎

4.5 Uniqueness

In this subsection we shall prove that the martingale problem for A, introduced in Section 3 as:

$$(4.38)\quad \begin{aligned} A = \Big\{ & \exp\{-\langle g_1, \mu_1\rangle + i\langle g_2, \mu_2\rangle\}, \\ & \exp\{-\langle g_1, \mu_1\rangle + i\langle g_2, \mu_2\rangle\} \left\langle \tfrac{1}{2}\left(-\Delta g_1 - 2\eta_1 g_1 + a_{11} g_1^2 - 2i a_{12} g_1 g_2 - a_{22} g_2^2\right), \mu_1 \right\rangle : \\ & g_1 \in \widetilde{\mathcal{D}}(\tfrac{1}{2}\Delta)_+,\ g_2 \in S(R^d)\Big\}, \end{aligned}$$

has a unique solution.

Throughout this article we worked with real-valued functions. Now for simplicity, we switch to complex-valued functions. We obtained in Corollary 4.12 that all the limit points of $\{(X^{(n)}, W^{(n)})\}$ are solutions of the martingale problem for (A_1, ν). By the obvious fact that

$$\Big\{ \exp\{-\langle g_1, \mu_1\rangle\} \sin(\langle g_2, \mu_2\rangle) \cup \exp\{-\langle g_1, \mu_1\rangle\} \cos(\langle g_2, \mu_2\rangle) : g_1 \in \mathcal{D}(A)_+, g_2 \in S \Big\} \subset \mathcal{D}(A_1),$$

we obtain that all the limit points are also solutions for (A, ν). Thus, in order to prove weak convergence, it is sufficient to prove uniqueness for (A, ν). Let (X, W) be any limit point of $\{(X^{(n)}, W^{(n)})\}$. For simplicity, in the next two lemmas we shall use the notation introduced in Section 4.4; that is $\langle g, W(t)\rangle$ and $\langle g, X(t)\rangle$ will be denoted by $W_t(g)$ and $X_t(g)$, respectively.

Lemma 4.18 *For each $g_1 \in \mathcal{D}(\frac{1}{2}\Delta)_+$, $g_2 \in S(R^d)$, the bracket process of $X_t(g_1)$ and $W_t(g_2)$ is*

$$(4.39)\qquad \langle X(g_1), W(g_2)\rangle_t = \int_0^t X_s(a_{12} g_1 g_2)\, ds.$$

Proof By Itô's formula

$$\begin{aligned} & \exp\{-X_t(g_1) + iW_t(g_2)\} = \\ & \exp\{-X_0(g_1) + iW_0(g_2)\} + \int_0^t \exp\{-X_s(g_1) + iW_s(g_2)\} X_s\left(-\eta_1 g_1 - \tfrac{1}{2}\Delta g_1\right) ds \\ & + \tfrac{1}{2}\int_0^t \exp\{-X_s(g_1) + iW_s(g_2)\}\, d(\langle -X(g_1) + iW(g_2)\rangle_s) + M_t, \end{aligned}$$

where M_t is, at least, a local martingale. Recall that

$$\begin{aligned} & \exp\{-X_t(g_1) + iW_t(g_2)\} \\ & - \tfrac{1}{2}\int_0^t \exp\{-X_s(g_1) + iW_s(g_2)\} X_s\left(-2\eta_1 g_1 - \Delta g_1 + a_{11}(g_1)^2 - 2i a_{12} g_1 g_2 - a_{22}(g_2)^2\right) ds \end{aligned}$$

is a martingale and

$$\langle X(g)\rangle_t = \int_0^t X_s\left(a_{11}g^2\right) ds, \qquad \forall g \in \mathcal{D}(\tfrac{1}{2}\Delta),$$

$$\langle W(g)\rangle_t = \int_0^t X_s\left(a_{22}g^2\right) ds, \qquad \forall g \in S(R^d).$$

Combining the above result with the fact that

$$\langle -X(g_1) + iW(g_2)\rangle_t = \langle X(g_1)\rangle_t - \langle W(g_2)\rangle_t - 2i\langle X(g_1), W(g_2)\rangle_t,$$

we are done. ∎

Let $C_0(R^d)$ be the set of continuous functions tending to zero at infinity. Introduce the additional notation

$$C'\left(R^d\right) = \left\{g : g = g_1 + ig_2 : g_1 \in C_l\left(R^d\right), g_2 \in C_0\left(R^d\right)\right\}, \tag{4.40}$$

$$C'\left(R^d\right)_+ = \left\{g : g \in C'\left(R^d\right), g_1 \in \tilde{\mathcal{D}}\left(\tfrac{1}{2}\Delta\right)_+, g_2 \in \mathcal{D}\left(\tfrac{1}{2}\Delta\right) \cap C_0\left(R^d\right)\right\}, \tag{4.41}$$

and the operator

$$\begin{aligned} \tilde{A} = \Big\{ & \exp\{-\langle g_1, \mu_1\rangle + i\langle g_2, \mu_2\rangle\}, \\ & \exp\{-\langle g_1, \mu_1\rangle + i\langle g_2, \mu_2\rangle\}\left\langle \tfrac{1}{2}\left(-\Delta g_1 - 2\eta_1 g_1 + a_{11}g_1^2 - 2ia_{12}g_1g_2 - a_{22}g_2^2\right), \mu_1\right\rangle : \\ & g_1 \in C'\left(R^d\right)_+, g_2 \in S\left(R^d\right)\Big\}. \end{aligned} \tag{4.42}$$

Then we have

Lemma 4.19 *(X, W) is a solution of the martingale problem for (A, ν) if, and only if, it is a solution of the martingale problem for $(\tilde{A}, \nu)$.*

Proof Since $A \subset \tilde{A}$, if (X, W) is a solution for $(\tilde{A}, \nu)$, then it is also a solution for (A, ν). Let (X, W) be a solution for (A, ν). For each $g_1 \in C'(R^d)_+$, $g_2 \in S(R^d)$, we obtain, by Itô's formula, that

$$\begin{aligned} & \exp\{-X_t(g_1) + iW_t(g_2)\} \\ = {} & \exp\{-X_0(g_1) + iW_0(g_2)\} + \int_0^t \exp\{-X_s(g_1) + iW_s(g_2)\}X_s\left(-\eta_1 g_1 - \tfrac{1}{2}\Delta g_1\right) ds \\ & + \tfrac{1}{2}\int_0^t \exp\{-X_s(g_1) + iW_s(g_2)\}\, d(\langle -X(g_1) + iW(g_2)\rangle_s) + M_t, \\ = {} & \exp\{-X_0(g_1) + iW_0(g_2)\} + \tfrac{1}{2}\int_0^t \exp\{-X_s(g_1) + iW_s(g_2)\} \\ & \times X_s\left(-2\eta_1 g_1 - \Delta g_1 + a_{11}(g_1)^2 - 2ia_{12}g_1g_2 - a_{22}(g_2)^2\right) ds + M_t, \end{aligned}$$

where the second equality follows by previous lemma, and M_t is, at least, a local martingale. But all the terms in the right and left hand sides are bounded, so that M_t is a martingale. ∎

In the proof of uniqueness the following lemma will play a crucial role:

Lemma 4.20 *The nonlinear evolution equation*

$$(4.43)\qquad \begin{cases} \dfrac{\partial U(t)}{\partial t} &= -\frac{1}{2}a_{11}U(t)^2 + U(t)\eta_1 + \frac{1}{2}\Delta U(t) + ia_{12}U(t)g_2 + \frac{1}{2}a_{22}(g_2)^2 \\ U(0) &= g_1 , \end{cases}$$

where $g_1 \in C'(R^d)_+$, $g_2 \in S(R^d)$, has a unique strong solution on R_+, such that $U(t)$, $\partial U(t)/\partial t$, and $\frac{1}{2}\Delta U(t)$ are continuous functions from R_+ to $C'(R^d)$, and $U(t) \in C'(R^d)_+$ for each $t \in R_+$.

Proof Appendix.

Lemma 4.21 *If $(X(t), W(t))$ is a solution of the martingale problem for $(\tilde{A}, \nu)$ in $D_{M_F\times S'}[0,\infty)$, then, for each $g_1 \in C'(R^d)_+$, $T > 0$,*

$$(4.44)\qquad \exp\{-\langle U(T-t), X(t)\rangle + i\langle g_2, W(t)\rangle\}$$

is a martingale for $0 \le t \le T$, where $U(t)$ is given by the unique solution of (4.43) with $U(0) = g_1 \in C'(R^d)_+$.

Proof Define

$$(4.45)\qquad F(g_1,g_2) = \eta_1 g_1 - \tfrac{1}{2}a_{11}(g_1)^2 + ia_{12}g_1g_2 + \tfrac{1}{2}a_{22}(g_2)^2 .$$

Let $\tilde{X}(t) \equiv (X(t), W(t))$ be a solution of the martingale problem for (A, ν), $\nu \in \mathcal{P}(M_F \times S')$. Define

$$(4.46)\qquad u\left(s, \tilde{X}(t)\right) = \exp\{-\langle U(T-s), X(t)\rangle + i\langle g_2, W(t)\rangle\},\ \forall\, 0 \le s < t \le T.$$

For each $(\mu_1, \mu_2) \in M_F \times S'$ we have

$$(4.47)\qquad \frac{\partial u(s,(\mu_1,\mu_2))}{\partial s} = \exp\{-\langle U(T-s),\mu_1\rangle + i\langle g_2,\mu_2\rangle\}\left\langle \left.\frac{\partial U(\bar{s})}{\partial \bar{s}}\right|_{\bar{s}=T-s}, \mu_1\right\rangle .$$

Hence

$$\begin{aligned} &E\left[u\left(t_2, \tilde{X}(t_2)\right) - u\left(t_1, \tilde{X}(t_2)\right)\Big|\mathcal{F}_{t_1}\right] \\ &= E\left[\int_{t_1}^{t_2} \exp\{-\langle U(T-s), X(t_2)\rangle + i\langle g_2, W(t_2)\rangle\}\left\langle \left.\frac{\partial U(\bar{s})}{\partial \bar{s}}\right|_{\bar{s}=T-s}, X(t_2)\right\rangle ds\,\Big|\mathcal{F}_{t_1}\right] \\ &\equiv E\left[\int_{t_1}^{t_2} v\left(s, \tilde{X}(t_2)\right) ds\,\Big|\mathcal{F}_{t_1}\right]. \end{aligned}$$

Fix $0 \le t_1 \le T$. Since $g_1 \in C'(R^d)_+$, we have $U(T-t_1) \in C'(R^d)_+$, for every $0 \le t_1 \le T$. Therefore, by the definition of the martingale problem for $(\tilde{A}, \nu)$, we have that

$$\begin{aligned} &\exp\{-\langle U(T-t_1)g_1, X(t)\rangle + i\langle g_2, W(t)\rangle\} \\ &\quad - \int_0^t \exp\{-\langle U(T-t_1), X(s)\rangle + i\langle g_2, W(s)\rangle\} \times \left\langle -F(U(T-t_1), g_2) - \tfrac{1}{2}\Delta U(T-t_1), X(s)\right\rangle ds \end{aligned}$$

is a martingale for $0 \le t \le T$. Define the function w by the following:

$$E\left[u\left(t_1, \tilde{X}(t_2)\right) - u\left(t_1, \tilde{X}(t_1)\right) \middle| \mathcal{F}_{t_1}\right] \tag{4.48}$$

$$= E\left[\exp\{-\langle U(T-t_1), X(t_2)\rangle + i\langle g_2, W(t_2)\rangle\}\right.$$

$$\left. - \exp\{-\langle U(T-t_1), X(t_1)\rangle + i\langle g_2, W(t_1)\rangle\} \middle| \mathcal{F}_{t_1}\right]$$

$$= E\left[\int_{t_1}^{t_2} \exp\{-\langle U(T-t_1), X(s)\rangle + i\langle g_2, W(s)\rangle\}\right.$$

$$\left. \times \left\langle -F(U(T-t_1), g_2) - \tfrac{1}{2}\Delta U(T-t_1), X(s)\right\rangle ds \middle| \mathcal{F}_{t_1}\right]$$

$$\equiv E\left[\int_{t_1}^{t_2} w\left(t_1, s, \tilde{X}(s)\right) ds \middle| \mathcal{F}_{t_1}\right], \ \forall\, 0 \le t_1 < t_2 \le T < \infty$$

It is easy to check that all conditions of the Theorem 4.3.4 [5] are satisfied and hence

$$u\left(t, \tilde{X}(t)\right) - \int_0^t \left\{v\left(s, \tilde{X}(s)\right) + w\left(s, s, \tilde{X}(s)\right)\right\} ds \tag{4.49}$$

is an $\mathcal{F}_t$-martingale. Finally, substituting the definitions of u, v, w we obtain the desired result. ∎

Theorem 4.22 *The martingale problem for (A, ν) has a unique solution.*

Proof Let $(X(t), W(t))$ be any solution of the martingale problem for (A, ν), $\nu \in \mathcal{P}(M_F \times S')$. Take $g_2 \in S(R^d)$ and $g_1 \in \tilde{\mathcal{D}}(\frac{1}{2}\Delta)_+$. (Recall this means that $g_1 = g_{11} + ig_{12} \in C(R^d)_+$, $g_{12} = 0$.) Then by the previous lemma, setting $T = t$, we obtain

$$E[\exp\{-\langle g_1, X(t)\rangle + i\langle g_2, W(t)\rangle\}] = E[\exp\{-\langle U(t), X(0)\rangle + i\langle g_2, W(0)\rangle\}], \ \forall\, 0 \le t < \infty,$$

where $U(t)$ is the unique solution of (4.43) with $U(0) = g_1$. Thus, by Corollary 1.9, we obtain that that any two solutions of the martingale problem for (A, ν) have the same one-dimensional distributions. By Theorem 1.4 the desired result follows. ∎

Remark 4.23 *The proof of Theorem 3.3 is now finished, since, by Corollary 4.12 and Theorem 4.22, all the conditions of Theorem 1.3 hold.*

Appendix: The solution of a particular non-linear evolution equation

Lemma A.1 *The nonlinear evolution equation*

$$(\pi) \quad \begin{cases} \dfrac{\partial U(t)}{\partial t} = -\frac{1}{2}a_{11}U(t)^2 + U(t)\eta_1 + \frac{1}{2}\Delta U(t) + ia_{12}U(t)g_2 + \frac{1}{2}a_{22}(g_2)^2 \\ U(0) = g_1, \end{cases}$$

where $g_1 \in C(R^d)_+$, $g_2 \in S(R^d)$, has a unique strong solution on R_+, such that $U(t)$, $\partial U(t)/\partial t$, $\frac{1}{2}\Delta U(t)$ are continuous functions from R_+ to $C(R^d)$, and $U(t) \in C(R^d)_+$ for each $t \in R_+$.

Recall that $C_l(R^d)_+$ is defined by (4.41).

Lemma A.2 *For each $g_1 \in C_l(R^d)_+$ there exists a unique strong solution of (π) on $[0, t_{max})$, where $t_{max} \leq \infty$. Moreover, if $t_{max} < \infty$, then $\lim_{t \uparrow t_{max}} \|U(t)\| = \infty$.*

Proof The proof was outlined by Perkins [12] for the case of real-valued functions. In our case the proof is completely the same (c.f. Theorems (3.13 — 3.15) from [12]). The more complicated part of existence of a solution without explosion will be given here. That is, we shall show that $t_{max} = \infty$.

Before starting our proofs we introduce several lemmas from [12].

Lemma A.3 *Let X^m be the super-Brownian motion given by the unique solution of martingale problem for $(\mathbf{A}, \delta_m)$, $m \in M_F$, where*

$$\mathbf{A} = \left\{ \exp\{-\langle g, \mu\rangle\},\ \exp\{-\langle g, \mu\rangle\} \left\langle -ag + \tfrac{1}{2}(\sigma g)^2 - \tfrac{1}{2}\Delta g, \mu \right\rangle : g \in \tilde{\mathcal{D}}\left(\tfrac{1}{2}\Delta\right)_+,\ a, \sigma \in R \right\}.$$

Then for each $\psi : [0, \infty) \times R^d \longrightarrow R$, such that $\psi(s)$, $\partial\psi(s)/\partial s$, $\frac{1}{2}\Delta\psi(s)$ are strongly continuous functions from $[0; T]$ to $C_l(R^d)$ we have

$$\text{(A.1)} \qquad X_t^m(\psi(t)) = m(\psi(0)) + \int_0^t X_s^m \left(\frac{\partial\psi(s)}{\partial s} + \tfrac{1}{2}\Delta\psi(s) + a\psi(s) \right) ds + Z_t(\psi),$$

where $Z_t(\psi)$ is the martingale with increasing process

$$\langle Z(\psi)\rangle_t = \int_0^t \sigma^2 X_s^m \left(\psi(s)^2\right) ds.$$

Lemma A.4 *The unique strong solution of the nonlinear evolution equation*

$$\begin{cases} \dfrac{\partial V(t)}{\partial t} &= -\tfrac{1}{2}\sigma^2 V(t)^2 + aV(t) + \tfrac{1}{2}\Delta V(t) \\ V(0) &= g \in \tilde{\mathcal{D}}(\tfrac{1}{2}\Delta)_+ \end{cases}$$

satisfies $V(t) \in C_l(R^d)_+$ for each $t \geq 0$.

Let

$$\text{(A.2)} \qquad U(t) = U_1(t) + iU_2(t)$$

be the unique strong solution of (π) on $[0, t_{max})$, where U_1 and U_2 are, respectively, the real and imaginary parts of the solution. We can see from (π) that that U_1 and U_2 satisfy

$$\text{(A.3)} \qquad \begin{cases} \dfrac{\partial U_1(t)}{\partial t} &= -\tfrac{1}{2}a_{11}U_1(t)^2 + U_1(t)\eta_1 + \tfrac{1}{2}\Delta U_1(t) \\ &\quad + \tfrac{1}{2}a_{11}U_2(t)^2 - a_{12}U_2(t)g_2 + \tfrac{1}{2}a_{22}(g_2)^2 \\ U_1(0) &= g_{11}, \end{cases}$$

$$\text{(A.4)} \qquad \begin{cases} \dfrac{\partial U_2(t)}{\partial t} &= -a_{11}U_1(t)U_2(t) + U_2(t)\eta_1 + \tfrac{1}{2}\Delta U_2(t) + a_{12}U_1(t)g_2 \\ U_2(0) &= g_{12}. \end{cases}$$

Let us define for each $h, f \in \overline{C}(R^d)$

$$F(h,f) = \tfrac{1}{2}a_{11}h^2 - a_{12}hf + \tfrac{1}{2}a_{22}f^2. \tag{A.5}$$

It is easy to check that

$$F(h,f)(x) \geq 0, \ \forall x \in R^d. \tag{A.6}$$

Lemma A.5 *Under the definitions (A.3), (A.4)* $U_1(t) \in C_l(R^d)_+$, *for all* $t < t_{max}$.

Proof We shall prove the result by contradiction. Assume that there exists a $t < t_{max}$, such that $\inf_{x \in R^d} U(t,x) \leq 0$. Define

$$t^* \equiv \inf \left\{ t < t_{max} : \inf_{x \in R^d} U(t,x) \leq 0 \right\}. \tag{A.7}$$

The solution $U(\cdot)$ of (π) is continuous on $[0, t_{max})$, and $g_1 \in C'(R^d)_+$. Thus $t^* > 0$, and, for each $s < t^*$,

$$U_1(s) \in C_l\left(R^d\right)_+. \tag{A.8}$$

Define $\psi(s) = U_1(t^* - s)$. Obviously we have

$$\frac{\partial \psi(s)}{\partial s} = - \left.\frac{\partial U_1(u)}{\partial u}\right|_{u=t^*-s}, \ \forall s \leq t^*.$$

Now let X^m be the super-Brownian motion which is the unique solution of the martingale problem for $(\mathbf{A}^*, \delta_m)$, $m \in M_F$, where

$$\mathbf{A}^* = \left\{ \exp\{-\langle g_{11}, \mu\rangle\},\ \exp\{-\langle g_{11},\mu\rangle\} \left\langle -\eta_1 g_{11} + \tfrac{1}{2}a_{11}(g_{11})^2 - \tfrac{1}{2}\Delta g_{11}, \mu \right\rangle : g_{11} \in \tilde{\mathcal{D}}\left(\tfrac{1}{2}\Delta\right)_+ \right\}.$$

By Lemma A.3, and equations (A.3), (A.5) we obtain, for all $t \leq t^*$,

$$\begin{aligned} & X_t^m\left(U_1(t^* - t)\right) \\ = \ & m(U_1(t^*)) + \int_0^t X_s^m \left(- \left.\frac{\partial U_1(u)}{\partial u}\right|_{u=t^*-s} + \tfrac{1}{2}\Delta U_1(t^*-s) + \eta_1 U_1(t^*-s) \right) ds + Z_s(\psi) \\ = \ & m(U_1(t^*)) + \int_0^t X_s^m \left(a_{11}U_1(t)^2\right) ds - \int_0^t X_s^m \left(F(U_2(T-s), g_2)\right) ds + Z_t(\psi), \end{aligned}$$

where $Z_t(\psi)$ is the martingale on $[0, t^*]$ with increasing process

$$\langle Z(\psi)\rangle_t = \int_0^t a_{11} X_s^m \left(U_1(t^*-s)^2\right) ds.$$

Rearanging, we obtain

$$-X_t^m\left(U_1(t^*-t)\right) - \int_0^t X_s^m\left(F(U_2(t^*-s), g_2)\right) ds = -m(U_1(t^*)) - Z_t(\psi) - \tfrac{1}{2}\langle Z(\psi)\rangle_t$$

and

$$\text{(A.9)} \qquad \exp\left\{-X_t^m\left(U_1(t^*-t)\right) - \int_0^t X_s^m\left(F(U_2(t^*-s), g_2)\right) ds\right\}$$
$$= \exp\left\{-m(U_1(t^*))\right\} \exp\left\{-Z_t(\psi) - \tfrac{1}{2}\langle Z(\psi)\rangle_t\right\}.$$

Note that $\exp\{-Z_t(\psi) - \frac{1}{2}\langle Z(\psi)\rangle_t\}$ is, at least, a local martingale on $[0, t^*]$, but that the left hand side of expression (A.9) is, by (A.6), (A.8), bounded and so is, in fact, a martingale. Setting $t = t^*$ we obtain

$$\text{(A.10)} \quad E\left[\exp\left\{-X_{t^*}^m(g_{11}) - \int_0^{t^*} X_s^m\left(F(U_2(t^*-s), g_2)\right) ds\right\}\right] = \exp\left\{-m(U_1(t^*))\right\}.$$

On the other hand, by (A.6),

$$\text{(A.11)} \quad E\left[\exp\left\{-X_{t^*}^m(g_{11}) - \int_0^{t^*} X_s^m\left(F(U_2(t^*-s), g_2)\right) ds\right\}\right] \leq E\left[\exp\left\{-X_{t^*}^m(g_{11})\, ds\right\}\right].$$

Define $V(t)$ to be the unique strong solution of

$$\begin{cases} \dfrac{\partial V(t)}{\partial t} = -\frac{1}{2}a_{11}V(t)^2 + \eta_1 V(t) + \frac{1}{2}\Delta V(t) \\ V(0) = g_{11} \in \tilde{\mathcal{D}}\left(\frac{1}{2}\Delta\right)_+. \end{cases}$$

By Lemma A.4 we have that $V(t) \in C_l(R^d)_+$ for each $t \geq 0$. By standard arguments from the theory of superprocess we find

$$\text{(A.12)} \qquad E\left[\exp\left\{-X_{t^*}^m(g_{11})\right\}\right] = \exp\left\{-m(V(t^*))\right\}.$$

Combining (A.10), (A.11), (A.12) we obtain that

$$\exp\left\{-m(U_1(t^*))\right\} \leq \exp\left\{-m(V(t^*))\right\}.$$

The above expression holds for all $m \in M_F$. Therefore, setting $m = \delta_x$, we obtain

$$\exp\left\{-U_1(t^*, x)\right\} \leq \exp\left\{-V(t^*, x)\right\}, \ \forall x \in R^d$$

and hence

$$\inf_{x\in R^d} U_1(t^*, x) \geq \inf_{x \in R^d} V(t^*, x).$$

But $\inf_{x\in R^d} V(t^*, x) > 0$, so that $\inf_{x\in R^d} U_1(t^*, x) > 0$, which is a contradiction. ∎

Lemma A.6 *There exist constants C_1, $C_2 \geq 0$ such that*

$$\text{(A.13)} \qquad \|U_2(t)\| \leq C_1 \exp\{C_2 t\}, \quad \forall t < t_{max}$$

and hence, if $t_{max} < \infty$, then

$$\text{(A.14)} \qquad \limsup_{t \uparrow t_{max}} \|U_2(t)\| \leq C_1 \exp\{C_2 t_{max}\}.$$

Proof Consider two cases

(1) $\eta_1 > 0$

Define

$$\tilde{U}_2(t) = U_2(t)\exp\{-\eta_1 t\}, \quad t < t_{max}. \tag{A.15}$$

By (A.4) we obtain

$$\begin{aligned}\frac{\partial \tilde{U}_2(t)}{\partial t} &= (-a_{11}U_1(t)U_2(t) + U_2(t)\eta_1 + \tfrac{1}{2}\Delta U_2(t) + a_{12}U_1(t)g_2)\exp\{-\eta_1 t\} \\ &\quad - \eta_1 U_2(t)\exp\{-\eta_1 t\} \\ &= -a_{11}U_1(t)\tilde{U}_2(t) + \tfrac{1}{2}\Delta\tilde{U}_2(t) + a_{12}U_1(t)g_2\exp\{-\eta_1 t\} \\ &= U_1(t)\left(-a_{11}\tilde{U}_2(t) + a_{12}g_2\exp\{-\eta_1 t\}\right) + \tfrac{1}{2}\Delta\tilde{U}_2(t), \quad t < t_{max}.\end{aligned}$$

Define

$$C_1 = 2\max\left(\|g_{12}\|, \left\|\frac{a_{22}g_2}{a_{11}}\right\|\right).$$

Assume that there exists $t_0 < t_{max}$, such that

$$\sup_{t \le t_0}\left\|\tilde{U}_2(t)\right\| > C_1.$$

Choose t^* such that

$$\left\|\tilde{U}_2(t^*)\right\| = \sup_{t \le t_0}\left\|\tilde{U}_2(t)\right\|. \tag{A.16}$$

By choice of C_1, we have that $t^* > 0$. For all $t < t_{max}$, $\tilde{U}_2(t) \in S(R^d)$, so that there exists a $x^* \in R^d$, for which

$$\left|\tilde{U}_2(t^*, x^*)\right| = \left\|\tilde{U}_2(t^*)\right\|.$$

Assume, without loss of generality, that x^* is the point of maximum of $\tilde{U}_2(t^*)$ (in the case of a minimum the proof is analogous). By the previous lemma, $U_1(t) \in C_l(R^d)_+$, for all $t < t_{max}$. By the choice of C_1 we obtain that

$$U_1(t^*, x^*)\left(-a_{11}\tilde{U}_2(t^*, x^*) + a_{12}g_2\exp\{-\eta_1 t\}\right) < 0.$$

Since x^* is the point of maximum of $\tilde{U}_2(t^*)$, the positive maximum principle ([5], p.165) implies that

$$\tfrac{1}{2}\Delta\tilde{U}_2(t^*, x^*) \le 0.$$

Hence

$$\frac{\partial \tilde{U}_2(t^*, x^*)}{\partial t^*} < 0.$$

Recall that $t^* > 0$, so that there exists a $\bar{t} < t^*$ for which $\tilde{U}_2(\bar{t}, x^*) > \tilde{U}_2(t^*, x^*)$. But this contradicts (A.16). Hence we obtain that

$$\sup_{t<t_{max}} \left\| \tilde{U}_2(t) \right\| \leq C_1,$$

and by the definition of $\tilde{U}_2$ we have

$$\|U_2(t)\| \leq C_1 \exp\{\eta_1 t\}, \ t < t_{max}, \tag{A.17}$$

which was what we wanted to prove.

(2) $\eta_1 \leq 0$

The proof is the same, the only difference being that we need not introduce $\tilde{U}_2$, as in the previous case. ∎

Proof of Lemma A.1 Assume by contradiction that $t_{max} < \infty$. It is well-known [11] that if $U(t)$ is a strong solution of (π) on $[0, t_{max})$ then it is also a mild solution of the following integral equation :

$$U(t) = S(t)g_1 + \int_0^t S(t-s)\left(-\tfrac{1}{2}a_{11}U(s)^2 + U(s)\eta_1 + ia_{12}U(s)g_2 + \tfrac{1}{2}a_{22}(g_2)^2\right) ds,$$

where $S(t)$ is the semigroup generated by $\frac{1}{2}\Delta$. By (A.2), (A.3) we obtain

$$U_1(t) = S(t)g_{11} + \int_0^t S(t-s)\left(-\tfrac{1}{2}a_{11}U_1(s)^2 + U_1(s)\eta_1 + F(U_2(s), g_2)\right) ds \tag{A.18}$$

Recall that, by Lemma A.5, we have $U_1(t) \in C(R^d)_+$ for all $t < t_{max}$. Thus

$$\begin{aligned} 0 < U_1(t) &= S(t)g_{11} + \int_0^t S(t-s)\left(-\tfrac{1}{2}a_{11}U_1(s)^2 + U_1(s)\eta_1\right) ds + \int_0^t S(t-s)F(U_2(s), g_2)\, ds \\ &\leq S(t)g_{11} + \int_0^t S(t-s)(U_1(s)\eta_1)\, ds + \int_0^t S(t-s)F(U_2(s), g_2)\, ds \\ &\leq \|g_{11}\| + |\eta_1| \int_0^t \|U_1(s)\|\, ds + t \sup_{s\leq t} \|F(U_2(s), g_2)\| \end{aligned}$$

Finally, defining $C = t_{max} \sup_{s \leq t_{max}} \|F(U_2(s), g_2)\|$, ($C < \infty$ by Lemma A.6) and $K = \|g_{11}\| + C$ we obtain

$$\|U_1(t)\| \leq K + |\eta_1| \int_0^t \|U_1(s)\|\, ds, \ \ \forall t < t_{max}, \tag{A.19}$$

and by Gronwall's inequality we have that

$$\|U_1(t)\| \leq K \exp\{|\eta_1| t\}, \ \ \forall t < t_{max}.$$

This implies that

$$\limsup_{t \uparrow t_{max}} \|U_1(t)\| \leq K \exp\{|\eta_1| t_{max}\} < \infty. \tag{A.20}$$

Combining the above result with Lemma A.6 we obtain

$$\lim_{t \uparrow t_{max}} \|U(t)\| \leq \limsup_{t \uparrow t_{max}} \|U_1(t)\| + \limsup_{t \uparrow t_{max}} \|U_2(t)\| < \infty, \tag{A.21}$$

which contradicts Lemma A.2. Hence $t_{max} = \infty$, which finishes the proof of the lemma. ∎

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